Kodama time

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Abstract

In a general (3+1)-dimensional spherically symmetric spacetime, a preferred time coordinate is introduced by applying the Clebsch decomposition theorem to the Kodama vector. Then a preferred coordinate system is constructed for the time-dependent metric tensor. However, certain ambiguities arise when the time-dependent metric is studied, such as the time-dependent redshift factor, and the time-dependent surface gravity. Nevertheless, by building a set of radial null geodesics, it is possible to define and calculate a notion of *bulk gravity* that generalizes the usual *surface gravity* and is valid throughout the entire spacetime geometry.

The Schwarzchild metric

The first non-trivial solution of the GR vacuum equations is given by the Schwarzchild metric [1]

$$\mathrm{d}s^2 = -\left(1 - \frac{2\,M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2\,M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2.$$

- Birkhoff's theorem assures the Schwarzchild metric is the only vacuum solution with spherical symmetry. Specifically there are no time-dependent solutions of this form.
- Although it seems to be a singularity at $2\,M = r$, a coordinate change shows the real singularity is at $r = 0$.
- Moreover, using different coordinate systems it is possible to study different aspects of the Schwarzchild solution.

Schwarzchild black holes

Three important aspects of such black holes are,

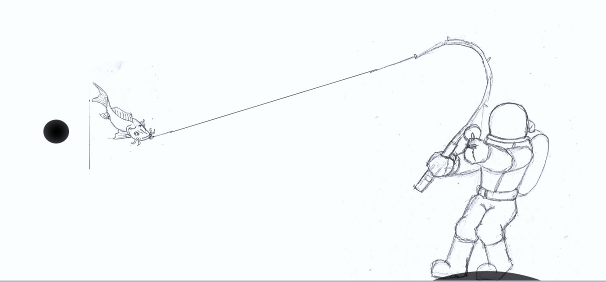
- **The redshift factor**: It is given by the g_{tt} component of the metric,

$$1 + z = \sqrt{1 - \frac{2\,M}{r}}.$$

- **The event horizon**: Is the spacelike surface where the Killing vector becomes null. It is located at

$$r = 2\,M.$$

Event horizons can not be *detected* locally.



- **The surface gravity**: It is *the gravitational acceleration of a stationary observer near the horizon, as seen at infinity*. For this metric it is

$$\kappa = \frac{1}{4M}.$$

These **three** factors depend on the notion of Killing vector.

- Static black holes are not the full story.
  - Black holes **evolve** until they reach an equilibrium between accretion and Hawking radiation.
  - Unfortunately, there are not many **solutions** of the Einstein equations for a **time-dependent metric**.
  - There is **no** time-like **Killing vector** for time-dependent spherically symmetric metrics.
  - This leaves notions (like the surface gravity, redshift factor and event horizon) which depend strongly on the existence of a Killing vector field, rather unclear.
- It's not all bad news.
  - Event horizons are not necessary to have Hawking radiation.
  - There are other ways to obtain a notion of surface gravity. Such as using the Kodama vector as some sort of substitute for the Killing vector [2], or through the **bulk gravity**.

Kodama vector

With spherical symmetry we have,

- **The time-dependent metric**:

$$\mathrm{d}s^2 = g_{ij}(x)\,\mathrm{d}x^i\mathrm{d}x^j + r(x)^2\mathrm{d}\Omega^2.$$

- **Kodama** proved [3] that the vector:

$$k^a = \epsilon_\perp^{ab}\,\nabla_b r.$$

  is divergence free.

- Furthermore, **Kodama's conservation law**:

$$\nabla_a S^a = \nabla_a(k_b\,G^{ab}) = 0.$$

Kodama's conservation law gives us an insight on the problem in spherical symmetry. Specifically,

- The Kodama vector can be decomposed as,

$$k^\flat = \alpha \, \mathrm{d}\beta \quad \rightarrow \quad k^\flat = F \, \mathrm{d}t.$$

Suggesting,

- A **natural** time coordinate t, which allows us to write the metric as,

$$\mathrm{d}s^2 = g_{tt} \, dt^2 + 2 g_{tr} dr \, dt + g_{rr} \, dr^2 + r^2 \mathrm{d}\Omega^2.$$

- And, since the Kodama vector is orthogonal to $\mathrm{d}r$,

$$\mathrm{d}s^2 = g_{tt} \, dt^2 + g_{rr} \, dr^2 + r^2 \mathrm{d}\Omega^2.$$

Finally, without loss of generality, we can write the metric as

$$ds^2 = -e^{-2\Phi(r,t)}\left(1 - \frac{2m(r,t)}{r}\right) dt^2 + \frac{dr^2}{1 - 2m(r,t)/r} + r^2 \mathrm{d}\Omega^2.$$

For this coordinate system, we have

1. The Kodama vector is $k^a = e^{\Phi}(1, 0, 0, 0)$.
2. The time translation vector T, does not coincide with the Kodama vector. In fact

$$e^{-2\Phi(r,t)} = \frac{||T||^2}{||k||^2}.$$

This expression allows us to foresee the ambiguity of the notion of *surface gravity*, for the dynamic case.

Additionally, with these coordinates it is clear that Kodama's conserved current implies,

$$\nabla_a S^a = \frac{2}{e^{-\Phi} r^2}\, \partial_a \left(-m', \dot{m}; 0, 0\right)^a = \frac{2}{e^{-\Phi} r^2} \left[-\dot{m}' + \dot{m}'\right] = 0.$$

Therefore, using the abstract definition of the Kodama vector, it can be proved the existence of an extra conserved quantity,

$$\nabla_a \left(\frac{\epsilon_{\perp}^{ab}}{r^2}\right) \equiv 0.$$

Both conservation laws can be written in the statement,

$$S^a_{new} = \frac{\epsilon_{\perp}^{ab}}{r^2}\, \nabla_b \Psi.$$

The ambiguous surface gravity

Two ways of calculating the surface gravity [1],

1. Using the Killing vector: $\kappa_s^2 = -\frac{1}{2}(\nabla_a \chi_b)(\nabla^a \chi^b)$.
2. Through the four-acceleration,

$$a = \sqrt{a_b\, a^b} = \sqrt{1 - \frac{2m(t,r)}{r}}\, \frac{\Phi'(t,r)}{r^2} + \frac{m'(t,r)}{r\sqrt{1-\frac{2m(t,r)}{r}}} - \frac{m}{r^2\sqrt{1-\frac{2m(t,r)}{r}}}.$$

- Both methods have problems when we study time-dependent geometries.

Alternatively, we can calculate the gravity throughout the whole spacetime, i.e. the *bulk gravity*. Consider the following null geodesics,

$$\ell_a = \frac{k_a + \nabla_a r}{e^{\Phi(t,r)}}, \quad n_a = \frac{k_a - \nabla_a r}{e^{\Phi(t,r)}}.$$

With, $\ell_a n^a = 2\, e^{-2\Phi(t,r)} ||k||^2$. They satisfy the geodesic equations,

$$\ell^b \nabla_b \ell^a = \kappa_\ell\, \ell^a\,, \quad n^b \nabla_b n^a = \kappa_n\, n^a.$$

From above we get, $\kappa_n + \kappa_\ell = -2\,\dot{\Phi}(t,r)$, and

$$\begin{aligned} \kappa_\ell &= \left(\frac{2\,m(t,r)}{r^2} - \frac{2\,m'(t,r)}{r}\right) e^{-\Phi(t,r)} \\ &- 2\left(1 - \frac{2\,m(t,r)}{r}\right) \Phi'(t,r)\, e^{-\Phi(t,r)} - \dot{\Phi}(t,r). \end{aligned}$$

Furthermore, this notion of gravitational acceleration is finite at $2\,m(t, r_H) = r_H$,

$$\kappa_H = \frac{1 - 2\,m'(t, r_H)}{r_H(t)} e^{-\Phi(t, r_H)} - \dot{\Phi}(t, r_H).$$

And, it coincides with the surface gravity in the static case. However, we have the freedom to normalize the null geodesics differently.

- With a different normalization the bulk gravity changes.
- The choice of normalization is quite arbitrary.

Trapping horizon

Although it is local, a trapping horizon [4] [5] guarantees Hawking radiation. To have a trapping horizon, the expansion of the null geodesics has to satisfy, at $r_H = 2\, m(t, r_H)$

1. $$\theta_\ell = 0.$$
2. $$\theta_n < 0.$$
3. $$n^a \nabla_a \theta_\ell < 0.$$

Let us use the following definition for the expansion [6] of the outward radial null geodesic,

$$\theta_\ell = \nabla_a \ell^a - \kappa_\ell,$$

and of the inward radial null geodesic,

$$\theta_n = \nabla_a n^a - \kappa_n.$$

To obtain

$$\theta_\ell = \frac{2}{r}\left(1 - \frac{2\,m(t,r)}{r}\right) e^{-\Phi(t,r)}.$$

and $\theta_n = -\theta_\ell$. Also,

$$(n^a \nabla_a \theta_\ell)_{r_H} = -\frac{4\,\dot{m}(t,r_H)}{r_H^2}\, e^{\Phi(t,r)}.$$

- We introduced a time coordinate parallel to the Kodama vector; the Kodama time.
- Then we constructed a coordinate system, for which:
  1. The outward and inward null geodesic gave us the notion of bulk gravity, which is valid through the whole geometry (except at $r = 0$).
  2. We have an apparent horizon at
  $$r_H = 2\,m(t, r_H).$$
  3. We have a trapping horizon, r_H, if
  $$\dot{m}(t, r_H) \geq 0.$$
- However, the freedom of normalization of the null geodesics produces an ambiguity on the notion of bulk gravity, and surface gravity. Nevertheless the horizon is still a trapping one.

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- Face the fact we have too much freedom!
- Work on the definition of quasi-local mass for this system of coordinates.
- Get a quasi-local notion of entropy.

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