Kodama time

Gabriel Abreu
Matt Visser

School of Mathematics, Statistics and Operation Research,
Victoria University of Wellington, New Zealand.

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Abstract

In a general (3+1)-dimensional spherically symmetric spacetime, a preferred time coordinate is introduced by applying the Clebsch decomposition theorem to the Kodama vector. Then a preferred coordinate system is constructed for the time-dependent metric tensor. However, certain ambiguities arise when the time-dependent metric is studied, such as the time-dependent redshift factor, and the time-dependent surface gravity. Nevertheless, by building a set of radial null geodesics, it is possible to define and calculate a notion of bulk gravity that generalizes the usual surface gravity and is valid throughout the entire spacetime geometry.
1. Introduction
   - The Schwarzchild solution

2. The Kodama miracle

3. Standard form of the metric
   - Kodama’s generalized conservation law

4. Bulk gravity
   - Dynamic surface gravity
   - Bulk gravity

5. The horizon

6. Discussion

7. References
The first non-trivial solution of the GR vacuum equations is given by the Schwarzschild metric [1]

\[ ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\Omega^2. \]

- Birkhoff’s theorem assures the Schwarzschild metric is the only vacuum solution with spherical symmetry. Specifically there are no time-dependent solutions of this form.
- Although it seems to be a singularity at \( 2M = r \), a coordinate change shows the real singularity is at \( r = 0 \).
- Moreover, using different coordinate systems it is possible to study different aspects of the Schwarzschild solution.
Schwarzschild black holes

Three important aspects of such black holes are,

- **The redshift factor**: It is given by the $g_{tt}$ component of the metric,

  $$1 + z = \sqrt{1 - \frac{2M}{r}}.$$ 

- **The event horizon**: Is the spacelike surface where the Killing vector becomes null. It is located at

  $$r = 2M.$$ 

Event horizons can not be *detected* locally.
The surface gravity: It is the gravitational acceleration of a stationary observer near the horizon, as seen at infinity. For this metric it is

\[ \kappa = \frac{1}{4M}. \]

These three factors depend on the notion of Killing vector.
Static black holes are not the full story.
- Black holes **evolve** until they reach an equilibrium between accretion and Hawking radiation.
- Unfortunately, there are not many solutions of the Einstein equations for a time-dependent metric.
- There is no time-like Killing vector for time-dependent spherically symmetric metrics.
- This leaves notions (like the surface gravity, redshift factor and event horizon) which depend strongly on the existence of a Killing vector field, rather unclear.

It’s not all bad news.
- Event horizons are not necessary to have Hawking radiation.
- There are other ways to obtain a notion of surface gravity. Such as using the Kodama vector as some sort of substitute for the Killing vector [2], or through the bulk gravity.
Kodama time

Kodama vector

With spherical symmetry we have,

- **The time-dependent metric:**

\[ ds^2 = g_{ij}(x) \, dx^i dx^j + r(x)^2 d\Omega^2. \]

- **Kodama** proved [3] that the vector:

\[ k^a = \epsilon_{ab}^{\perp} \nabla_b r. \]

is divergence free.

- Furthermore, **Kodama’s conservation law:**

\[ \nabla_a S^a = \nabla_a (k_b \, G^{ab}) = 0. \]
Kodama’s conservation law gives us an insight on the problem in spherical symmetry. Specifically,

- The Kodama vector can be decomposed as,

\[ k^\flat = \alpha \, d\beta \quad \rightarrow \quad k^\flat = F \, dt. \]

Suggesting,

- A **natural** time coordinate \( t \), which allows us to write the metric as,

\[ ds^2 = g_{tt} \, dt^2 + 2g_{tr} \, dr \, dt + g_{rr} \, dr^2 + r^2 d\Omega^2. \]

- And, since the Kodama vector is orthogonal to \( dr \),

\[ ds^2 = g_{tt} \, dt^2 + g_{rr} \, dr^2 + r^2 d\Omega^2. \]
Finally, without loss of generality, we can write the metric as

$$ds^2 = -e^{-2\Phi(r,t)} \left(1 - \frac{2m(r,t)}{r}\right) dt^2 + \frac{dr^2}{1 - 2m(r,t)/r} + r^2 d\Omega^2.$$  

For this coordinate system, we have

1. The Kodama vector is $k^a = e^\Phi(1, 0, 0, 0)$.
2. The time translation vector $T$, does not coincide with the Kodama vector. In fact

$$e^{-2\Phi(r,t)} = \frac{||T||^2}{||k||^2}.$$  

This expression allows us to foresee the ambiguity of the notion of surface gravity, for the dynamic case.
Additionally, with these coordinates it is clear that Kodama’s conserved current implies,

\[ \nabla_a S^a = \frac{2}{e^{-\Phi} r^2} \partial_a (-m', \dot{m}; 0, 0)^a = \frac{2}{e^{-\Phi} r^2} [-\dot{m}' + \dot{m}'] = 0. \]

Therefore, using the abstract definition of the Kodama vector, it can be proved the existence of an extra conserved quantity,

\[ \nabla_a \left( \frac{\epsilon_{ab}}{r^2} \right) \equiv 0. \]

Both conservation laws can be written in the statement,

\[ S_{new}^a = \frac{\epsilon_{ab}}{r^2} \nabla_b \Psi. \]
The ambiguous surface gravity

Two ways of calculating the surface gravity \([1]\),

1. Using the Killing vector:
   \[ \kappa_s^2 = -\frac{1}{2} (\nabla_a \chi_b)(\nabla^a \chi^b). \]

2. Through the four-acceleration,
   \[
   a = \sqrt{a_b a^b} = \sqrt{1 - \frac{2m(t, r)}{r} \frac{\Phi'(t, r)}{r^2}} + \frac{m'(t, r)}{r \sqrt{1 - \frac{2m(t, r)}{r}}} - \frac{m}{r^2 \sqrt{1 - \frac{2m(t, r)}{r}}}. 
   \]

Both methods have problems when we study time-dependent geometries.
Alternatively, we can calculate the gravity throughout the whole spacetime, i.e. the *bulk gravity*. Consider the following null geodesics,

\[ \ell_a = \left( k_a + \nabla_a r \right) e^{\Phi(t,r)}, \quad n_a = \left( k_a - \nabla_a r \right) e^{\Phi(t,r)}. \]

With, \( \ell_a n^a = 2 e^{-2\Phi(t,r)} \|k\|^2 \). They satisfy the geodesic equations,

\[ \ell^b \nabla_b \ell^a = \kappa_\ell \ell^a, \quad n^b \nabla_b n^a = \kappa_n n^a. \]

From above we get, \( \kappa_n + \kappa_\ell = -2 \dot{\Phi}(t, r) \), and

\[ \kappa_\ell = \left( \frac{2 m(t, r)}{r^2} - \frac{2 m'(t, r)}{r} \right) e^{-\Phi(t,r)} - 2 \left( 1 - \frac{2 m(t, r)}{r} \right) \Phi'(t, r) e^{-\Phi(t,r)} - \dot{\Phi}(t, r). \]
Furthermore, this notion of gravitational acceleration is finite at $2 \, m(t, r_H) = r_H$,

$$\kappa_H = \frac{1 - 2 \, m'(t, r_H)}{r_H(t)} \, e^{-\Phi(t, r_H)} - \dot{\Phi}(t, r_H).$$

And, it coincides with the surface gravity in the static case. However, we have the freedom to normalize the null geodesics differently.

- With a different normalization the bulk gravity changes.
- The choice of normalization is quite arbitrary.
Trapping horizon

Although it is local, a trapping horizon [4] [5] guarantees Hawking radiation. To have a trapping horizon, the expansion of the null geodesics has to satisfy, at $r_H = 2 m(t, r_H)$

1. $\theta_\ell = 0.$

2. $\theta_n < 0.$

3. $n^a \nabla_a \theta_\ell < 0.$
Let us use the following definition for the expansion [6] of the outward radial null geodesic,

$$\theta_\ell = \nabla_a \ell^a - \kappa_\ell,$$

and of the inward radial null geodesic,

$$\theta_n = \nabla_a n^a - \kappa_n.$$

To obtain

$$\theta_\ell = \frac{2}{r} \left( 1 - \frac{2m(t, r)}{r} \right) e^{-\Phi(t, r)}.$$

and $\theta_n = -\theta_\ell$. Also,

$$(n^a \nabla_a \theta_\ell)_{r_H} = -\frac{4 \dot{m}(t, r_H)}{r_H^2} e^{\Phi(t, r)}.$$
We introduced a time coordinate parallel to the Kodama vector; the Kodama time.

Then we constructed a coordinate system, for which:

1. The outward and inward null geodesic gave us the notion of bulk gravity, which is valid through the whole geometry (except at $r = 0$).

2. We have an apparent horizon at

$$r_H = 2 \, m(t, r_H).$$

3. We have a trapping horizon, $r_H$, if

$$\dot{m}(t, r_H) \geq 0.$$

However, the freedom of normalization of the null geodesics produces an ambiguity on the notion of bulk gravity, and surface gravity. Nevertheless the horizon is still a trapping one.
What’s next?

- Face the fact we have too much freedom!
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■ Work on the definition of quasi-local mass for this system of coordinates.
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- Face the fact we have too much freedom!
- Work on the definition of quasi-local mass for this system of coordinates.
- Get a quasi-local notion of entropy.
References


