A Topology for the Abstract Boundary Construction for Space-time

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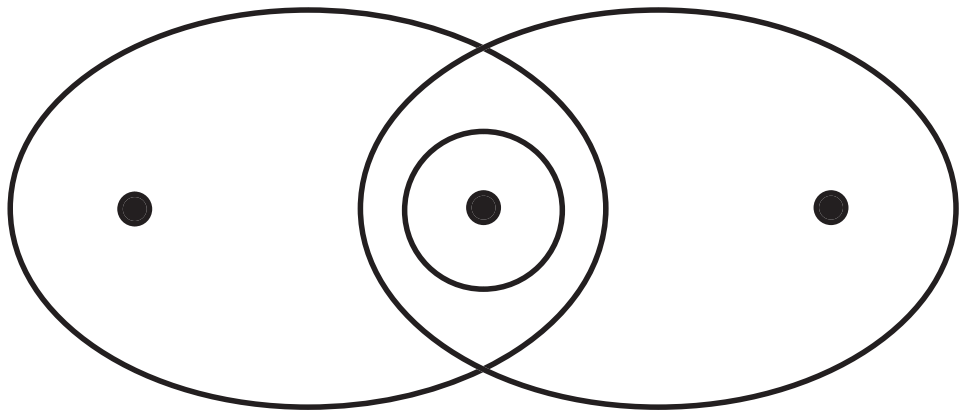
Boundary Construction

- A boundary construction is a mathematical tool for locating and classifying the boundary points of a manifold. Interesting things occur at the boundary - singularities and infinities, for example.
- The abstract boundary construction[1] is one of a handful of boundary constructions.
  - b–boundary
  - c–boundary
  - g-boundary

[1] Scott, S. M. and Szekeres, P. (1994), 'The Abstract Boundary - A New Approach to Singularities of Manifolds', *J.Geom.Phys.* **13**, 223-253.

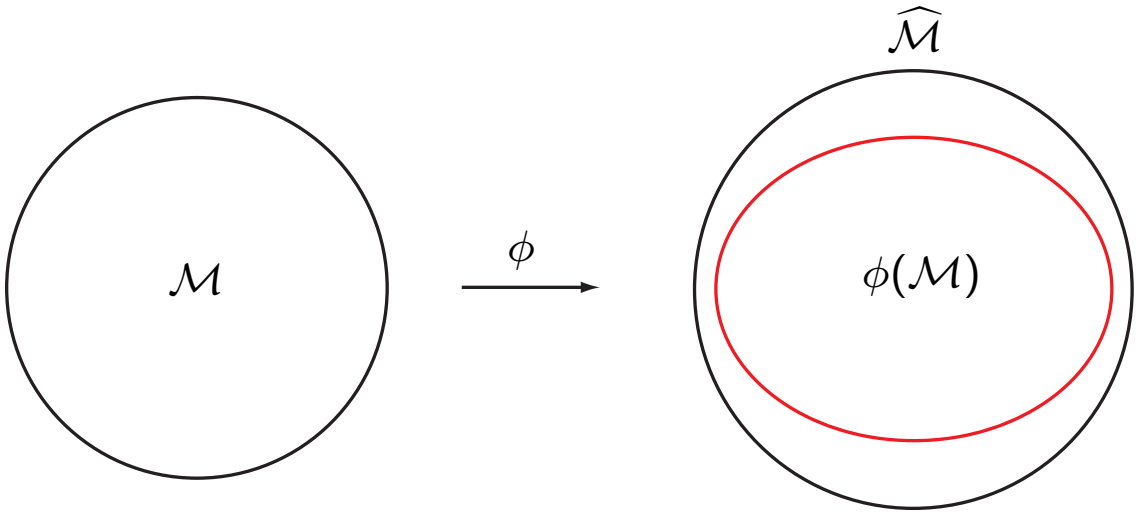
Topology

- A topology tells us how the abstract boundary points are connected to the rest of the manifold.
- Provides us with a full framework in which to describe the singularities of a manifold.
- Specifically, a topology on a space X is a collection of sets of elements of X.
- These sets obey certain rules and are called open sets.



Embedding

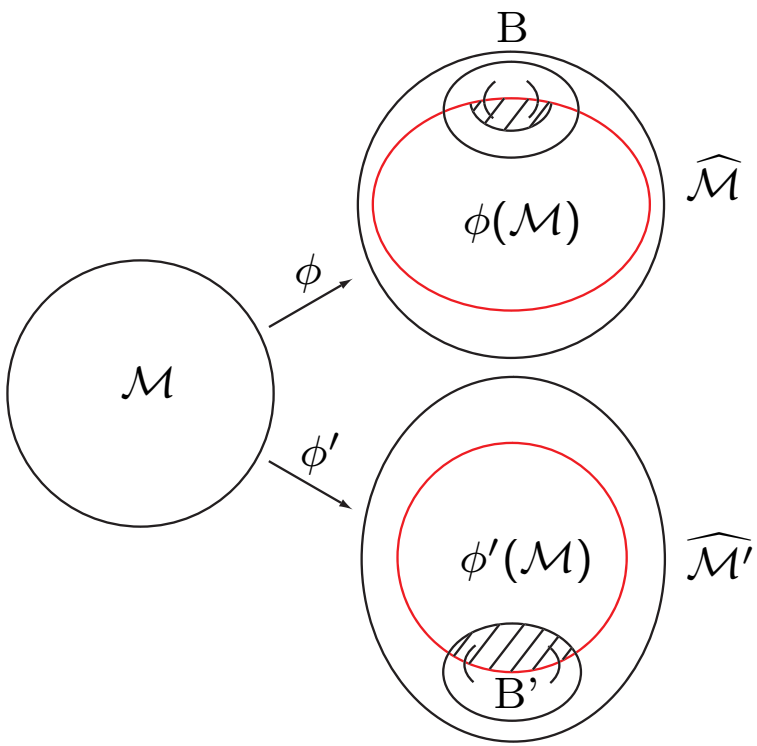
- A boundary for an n-dimensional manifold is obtained by embedding that manifold into a larger n-dimensional manifold.



- The boundary points are the limit points of $\phi(\mathcal{M})$ in $\widehat{\mathcal{M}}$

Covering Relation

- There are an infinite number of different embeddings.
- The manifold may appear to be singular with respect to one embedding and non-singular with respect to another.
- The abstract boundary considers all possible embeddings.

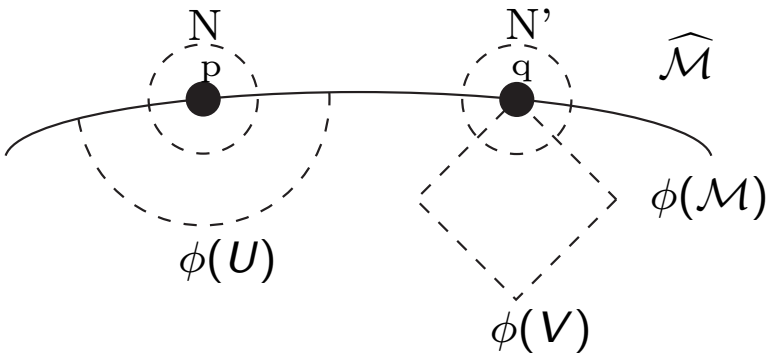


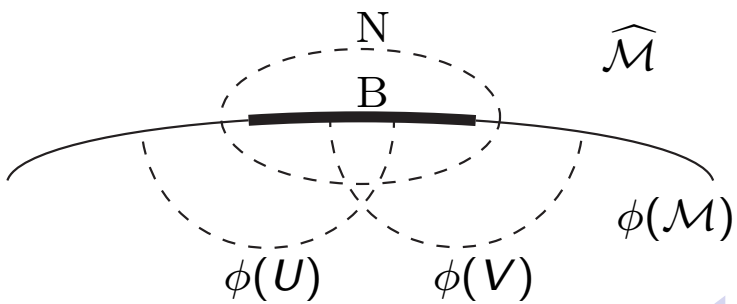
Abstract Boundary Point

- The covering relation will give collections of sets that are equivalent to each other: $B \sim B' \sim B''$... (abstract boundary sets)
- An abstract boundary point is an abstract boundary set that has a singleton point p as a representative member. Denote the abstract boundary point by $[p]$.

Attached Point

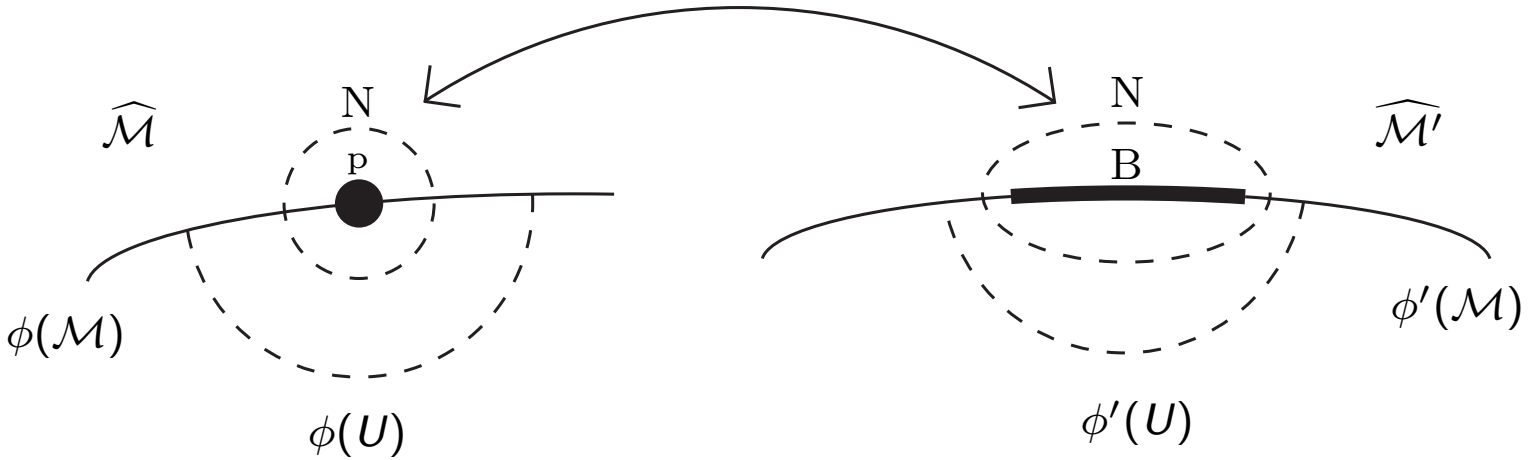
- Every manifold has a topology defined upon it by its chart structure: the atlas topology.
- Consider abstract boundary points to be 'attached' to the open sets of this topology.





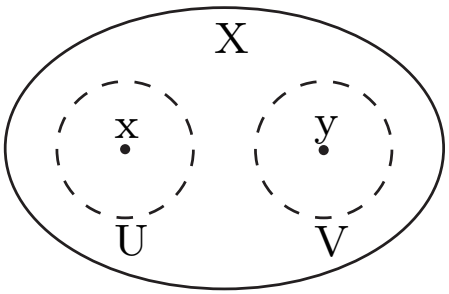
Attached Abstract Boundary Point

- An abstract boundary point $[p]$ is attached if the abstract boundary point representative p is attached.

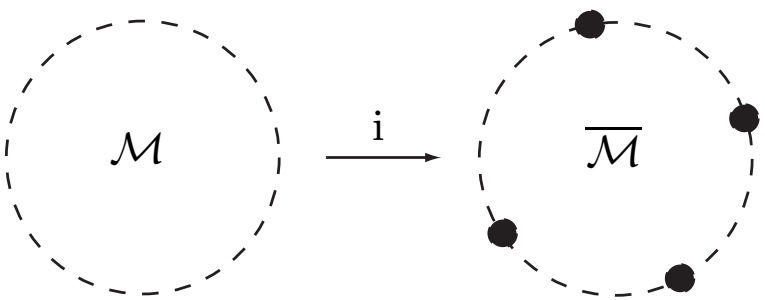


Attached Point Topology Results

- The attached point topology is Hausdorff: every Cauchy sequence in the space converges to one limit point.

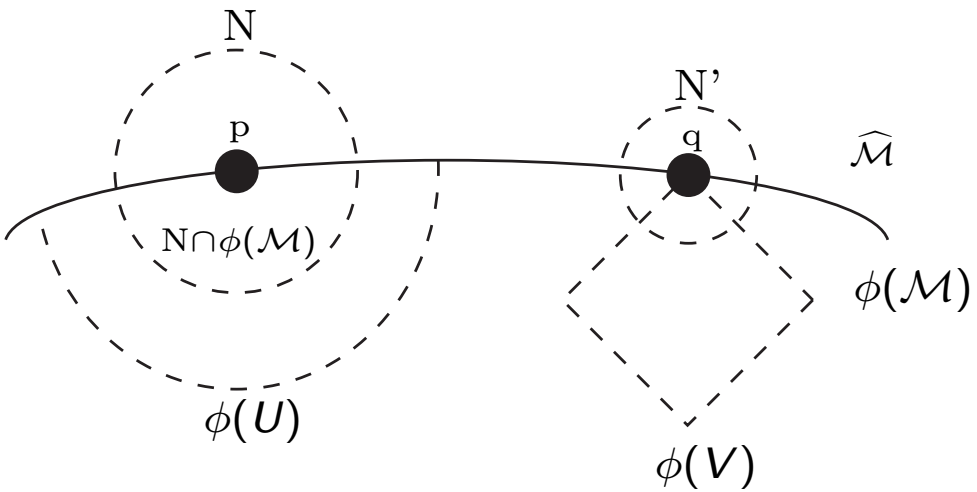


- The mapping $i : \mathcal{M} \to \overline{\mathcal{M}}$ is an embedding.



Strongly Attached Point

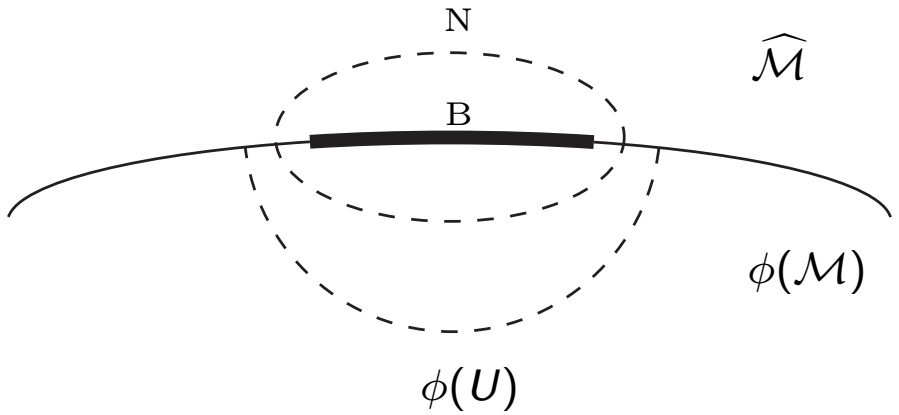
- The strongly attached point topology is defined similarly to the attached point topology.
- The open sets of $\mathcal{M}$ to which abstract boundary points can be attached are different.



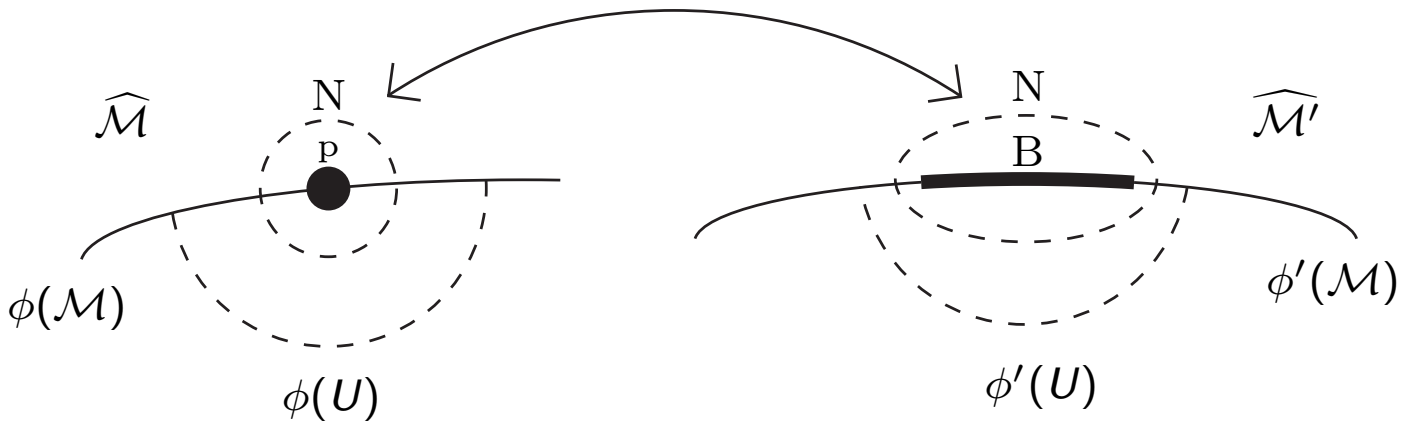
- This removes the occurrence of boundary points being attached to the 'wedge' shaped open sets of $\mathcal{M}$.

Strongly Attached Abstract Boundary Point

B strongly attached to U.

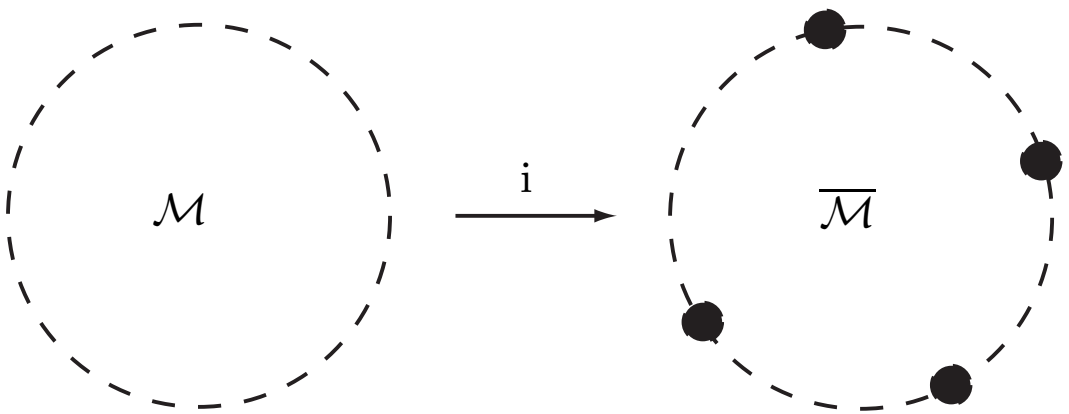


$[p]$ strongly attached to U.



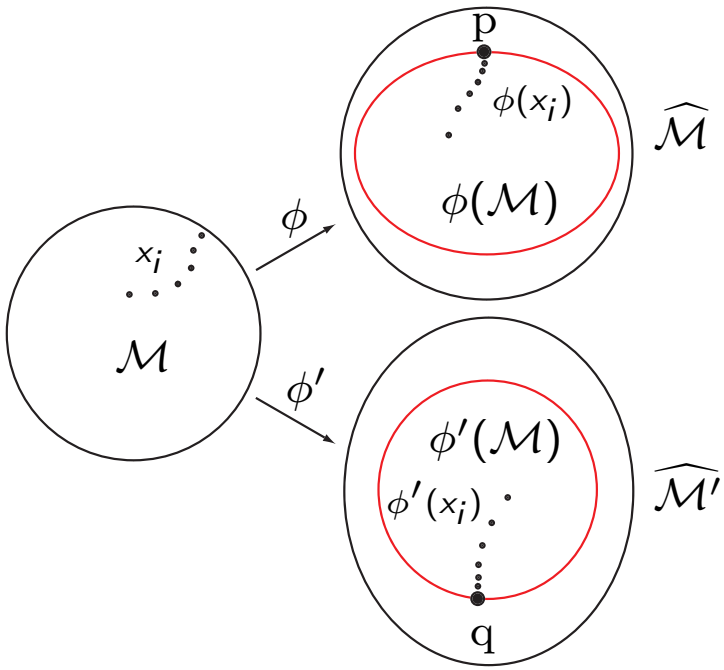
Strongly Attached Point Topology Results

- The strongly attached point topology is not Hausdorff: sequences can have more than one limit point.
- The mapping $i : \mathcal{M} \to \overline{\mathcal{M}}$ is an embedding.



Separation of Boundary Points

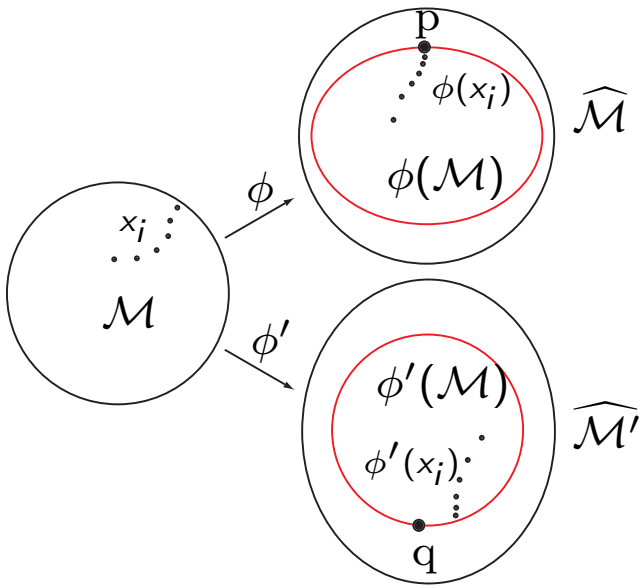
- Boundary points of one embedding may be *in contact* with boundary points of another embedding.



- Hausdorff separation is lost between abstract boundary points that are in contact.

Partial Cross Sections

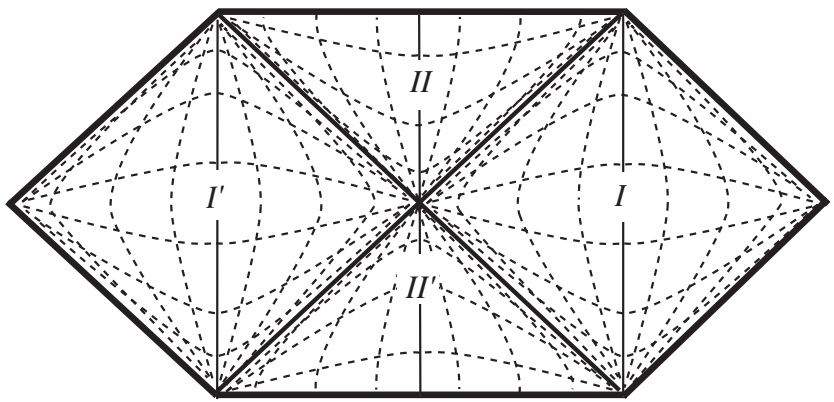
- A partial cross section is a collection of abstract boundary points which are not in contact.



- A partial cross section of a space-time is defined by each embedding, $\sigma_\phi := \{[p] \mid p \in \partial_\phi \mathcal{M}\}$.
- In effect, partial cross sections generalise the notion of an embedding.

Optimal Embeddings

- In an optimal embedding, the important features of a space-time, i.e., singularities and points at infinity, are made apparent.[2]



- We can define an optimal embedding to be a 'maximal' partial cross section which clearly displays the physical properties of the original space-time.

[2] Penrose-Carter diagram donated by Phil Threlfall

Summary

- The strongly attached topology describes the contact properties of the abstract boundary.
- These contact properties are important for determining partial cross sections of a space-time.
- Partial cross sections are used to build optimal embeddings, which clearly display the important features of a space-time.

Bibliography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