# SU(2) Hamiltonian structure in the Holst formulation of gravity.

**Francesco Cianfrani**, in collaboration with Giovanni Montani

Università di Roma "Sapienza" - ICRA

$5^{th}$ Australasian Conference on General Relativity and Gravitation,
Christchurch, December $16^{th}$ 2009

# Introduction

Fundamental interactions find an ultimate description in the framework of Quantum Field Theory for Yang-Mills models. Hence, the combination of known quantization techniques with gauge invariance leads to consistent theoretical models and, more important, to a proper description of the physical reality.

Many attempts towards Quantum Gravity are based on applying standard quantization procedures and on looking for a classical formulation of General Relativity close to the one of gauge theories.

For instance, in Loop Quantum Gravity geometric degrees of freedom are described via real SU(2) connections (Ashtekar-Barbero-Immirzi connections), which arise when the local Lorentz frame is fixed according with the time-gauge condition.

Here the underlying classical (Holst) formulation is considered and it is outlined that the associated Hamiltonian structure resembles that one of a background-independent SU(2) gauge theory no matter the local Lorentz frame is fixed or not. Then it will be presented the application of this scheme to some cases when matter fields are present.

# The Holst formulation

Let us consider a space-time manifold endowed with a metric tensor $g_{\mu\nu}$ and fix the local Lorentz frame by 4-bein vectors $e^a_\mu$ ($g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$, $\eta_{ab} = diag[1,-1,-1,-1]$) and spin connections $\omega^{AB}_\mu$.

The Holst formulation for gravity is based on the action

$$S = \int \sqrt{-g} \left[ e^\mu_A e^\nu_B R^{AB}_{\mu\nu} - \frac{1}{2\gamma} \epsilon^{AB}{}_{CD} e^\mu_A e^\nu_B R^{CD}_{\mu\nu} \right] d^4x,$$

$$R^{AB}_{\mu\nu} = \partial_{[\mu} \omega^{AB}_{\nu]} - \omega^A{}_{C[\mu} \omega^{CB}_{\nu]}, \qquad \gamma \text{ Immirzi parameter.}$$

The additional term with respect to the Einstein-Hilbert action does not provide any classical modification (it vanishes "on-shell").

In the following 4-bein components will be denoted as follows

$$e^0_\mu = (N, -\chi_a E^a_i), \qquad e^a_\mu = (E^a_i N^i, E^a_i).$$

1)*Hamiltonian analysis:*

Conjugate momenta: ${}^{\gamma}\pi^i_{AB} = \pi^i_{AB} - \frac{1}{2\gamma}\epsilon^{CD}{}_{AB}\pi^i_{CD}, \qquad \pi^i_{AB} = 2\sqrt{-g}\,e^t_{[A}e^i_{B]}.$

$$\mathcal{H} = \int \left[ \frac{\widetilde{N}}{\sqrt{h}} H + \widetilde{N}^i H_i - \left(\omega_t^{AB} - \frac{1}{2\gamma}\omega_t^{CD}\epsilon^{AB}{}_{CD}\right) G_{AB} + \lambda_{ij}C^{ij} + \eta_{ij}D^{ij} + \lambda^{AB}\pi^t_{AB}\right] d^3x.$$

## Hamiltonian constraints

$$\begin{cases} H = \pi^i_{AF}\pi^{jF}{}_B\left(R^{AB}_{ij} - \frac{1}{2\gamma}\epsilon^{AB}{}_{CD}R^{CD}_{ij}\right) = 0, \mathrm{super-Hamiltonian} \\ H_i = \pi^j_{AB}\left(R^{AB}_{ij} - \frac{1}{2\gamma}\epsilon^{AB}{}_{CD}R^{CD}_{ij}\right) = 0, \mathrm{super-momentum} \\ G_{AB} = \partial_i\pi^i_{AB} - 2\omega_{[A}{}^C{}_i\pi^i_{|C|B]} = 0, \mathrm{Lorentz\ \ constraint} \\ \left.\begin{array}{l} C^{ij} = \epsilon^{ABCD}\pi^{(i}_{AB}\pi^{j)}_{CD} = 0, \\ D^{ij} = \epsilon^{ABCD}\pi^k_{AF}\pi^{(iF}{}_B D_k\pi^{j)}_{CD} = 0, \end{array}\right\} 2^{\mathrm{nd}}-\mathrm{class\ constraints} \end{cases} .$$

The set of constraints is second-class, which means that some variables are redundant and a non-trivial symplectic structure is inferred on the constraint hypersurfaces.

## 2) *Solutions of* $2^{nd}$ *class constraints:*

Geometric structures ($\pi^i_{0b} = \pi^i_b$):

$$\pi^i_a = \frac{\sqrt{h}}{\sqrt{1+\chi^2}} E^i_a, \qquad h_{ij} = \frac{1}{\pi} T^{-1}_{ab} \pi^a_i \pi^b_j \qquad T^{-1}_{ab} = \eta_{ab} + \chi_a \chi_b$$

$$^{\pi}\omega_a{}^b{}_i = E^{b}_j {}^3\nabla_i E^j_a = \frac{1}{\pi^{1/2}} \pi^{b}_l {}^3\nabla_i (\pi^{1/2} \pi^l_a) \qquad (\pi^a_i \pi^i_b = \delta^a_b), \qquad {}^{\pi}D_i \chi_a = \partial_i \chi_a - {}^{\pi}\omega_a{}^b{}_i \chi_b.$$

The solution of $C^{ij} = 0$ and $D^{kl} = 0$ is given by:

$$\omega_a{}^b{}_i = {}^{\pi}\omega_a{}^b{}_i + \chi_a \omega^{0b} + \chi^b (\omega_a{}^0{}_i - {}^{\pi}D_i \chi_a), \qquad \pi^i_{ab} = 2\chi_{[a} \pi^i_{b]}$$

**$\chi_a$ are promoted to configuration variables, such that no gauge fixing of the local Lorentz frame occurs, while second-class constraints are solved.**

2)*Solutions of* $2^{nd}$ *class constraints:*

Geometric structures ($\pi^i_{0b} = \pi^i_b$):

$$\pi^i_a = \frac{\sqrt{h}}{\sqrt{1+\chi^2}} E^i_a, \qquad h_{ij} = \frac{1}{\pi} T^{-1}_{ab} \pi^a_i \pi^b_j \qquad T^{-1}_{ab} = \eta_{ab} + \chi_a \chi_b$$

$${}^\pi\omega_a{}^b{}_i = E^{b}_j{}^3\nabla_i E^j_a = \frac{1}{\pi^{1/2}} \pi^{b}_l{}^3\nabla_i(\pi^{1/2}\pi^l_a) \qquad (\pi^a_i \pi^i_b = \delta^a_b), \qquad {}^\pi D_i \chi_a = \partial_i \chi_a - {}^\pi\omega_a{}^b{}_i \chi_b.$$

The solution of $C^{ij} = 0$ and $D^{kl} = 0$ is given by:

$$\omega_a{}^b{}_i = {}^\pi\omega_a{}^b{}_i + \chi_a \omega^{0b} + \chi^b(\omega_a{}^0{}_i - {}^\pi D_i \chi_a), \qquad \pi^i_{ab} = 2\chi_{[a}\pi^i_{b]}$$

$\chi_a$ **are promoted to configuration variables, such that no gauge fixing of the local Lorentz frame occurs, while second-class constraints are solved.**

Induced symplectic form:

$$\{\pi^i_a(x,t),\pi^j_b(y,t)\}=0$$

$$\{\omega^{0a}_i(x,t),\omega^{0b}_j(y,t)\}=$$
$$=\left(-\frac{1}{2\gamma(1+\chi^2)^2}T^{-1a}_{\ \ c}T^{-1b}_{\ \ d}T^{-1g}_{\ \ h}\epsilon^{d}_{\ fg}-\frac{1}{1+\chi^2}T^{-1a}_{\ \ c}\chi_h\delta^b_f\right)\frac{\partial^\pi\omega^{fh}_j(y,t)}{\partial\pi^i_c(x,t)}-$$
$$-\left(-\frac{1}{2\gamma(1+\chi^2)^2}T^{-1b}_{\ \ c}T^{-1a}_{\ \ d}T^{-1g}_{\ \ h}\epsilon^{d}_{\ fg}-\frac{1}{1+\chi^2}T^{-1b}_{\ \ c}\chi_h\delta^a_f\right)\frac{\partial^\pi\omega^{fh}_i(x,t)}{\partial\pi^j_c(y,t)}$$

$$\{\omega^{0a}_i(x,t),\pi^j_b(y,t)\}=\delta^j_i\delta^3(x-y)\frac{1}{1+\chi^2}T^{-1a}_{\ \ b}.$$

3)*SU(2) gauge structure:*

The adopted solutions of $C^{ij} = D^{kl} = 0$ allow to parametrize the constraints hypersurfaces by the two sets of conjugate variables $\{{}^{(\gamma)}\widetilde{A}^a_i, {}^{(\gamma)}\widetilde{\pi}^j_b\}$ and $\{\chi_c, \pi^d\}$

- ${}^{(\gamma)}\widetilde{\pi}^i_a = \frac{1}{\gamma}\widetilde{\pi}^i_a = \frac{1}{\gamma}S^b_a\pi^i_b$ ($\widetilde{\pi}^i_a$ 3-bein densitized vectors),
  $S^a_b = \sqrt{1+\chi^2}\delta^a_b + \frac{1-\sqrt{1+\chi^2}}{\chi^2}\chi_a\chi_b$
- ${}^{(\gamma)}\widetilde{A}^a_i =$
  $S^{-1a}_b\left(\gamma(1+\chi^2)T^{bc}\left(\omega_{0ci} + {}^{\pi}D_i\chi_c\right) - \frac{1}{2}\epsilon^b{}_{cd}{}^{\pi}\omega^{cf}{}_i T^{-1d}_f + \frac{2+\chi^2-2\sqrt{1+\chi^2}}{2\chi^2}\epsilon^{abc}\partial_i\chi_b\chi_c\right)$

From $G_{AB} = 0$ the following constraints are obtained

$$G_a = \partial_i{}^{(\gamma)}\widetilde{\pi}^i_a + \epsilon_{abc}{}^{(\gamma)}\widetilde{A}^b_i{}^{(\gamma)}\widetilde{\pi}^i_c = 0. \qquad \pi^a = 0.$$

- The SU(2) gauge structure arises also when the time-gauge condition is relaxed. (${}^{(\gamma)}\widetilde{A}^a_i$ are GENERALIZED ASHTEKAR-BARBERO-IMMIRZI CONNECTIONS);
- $\chi_a$ are non-dynamical variables.

3)*SU(2) gauge structure:*

The adopted solutions of $C^{ij} = D^{kl} = 0$ allow to parametrize the constraints hypersurfaces by the two sets of conjugate variables $\{{}^{(\gamma)}\widetilde{A}^a_i, {}^{(\gamma)}\widetilde{\pi}^j_b\}$ and $\{\chi_c, \pi^d\}$

- ${}^{(\gamma)}\widetilde{\pi}^i_a = \frac{1}{\gamma}\widetilde{\pi}^i_a = \frac{1}{\gamma}S^b_a\pi^i_b$ ($\widetilde{\pi}^i_a$ 3-bein densitized vectors),
  $S^a_b = \sqrt{1+\chi^2}\delta^a_b + \frac{1-\sqrt{1+\chi^2}}{\chi^2}\chi_a\chi_b$
- ${}^{(\gamma)}\widetilde{A}^a_i =$
  $S^{-1a}_b\left(\gamma(1+\chi^2)T^{bc}\left(\omega_{0ci} + {}^{\pi}D_i\chi_c\right) - \frac{1}{2}\epsilon^b{}_{cd}{}^{\pi}\omega^{cf}{}_i T^{-1d}_f + \frac{2+\chi^2-2\sqrt{1+\chi^2}}{2\chi^2}\epsilon^{abc}\partial_i\chi_b\chi_c\right)$

From $G_{AB} = 0$ the following constraints are obtained

$$G_a = \partial_i{}^{(\gamma)}\widetilde{\pi}^i_a + \epsilon_{abc}{}^{(\gamma)}\widetilde{A}^{b(\gamma)}_i\widetilde{\pi}^i_c = 0. \qquad \pi^a = 0.$$

- The SU(2) gauge structure arises also when the time-gauge condition is relaxed. (${}^{(\gamma)}\widetilde{A}^a_i$ are GENERALIZED ASHTEKAR-BARBERO-IMMIRZI CONNECTIONS);
- $\chi_a$ are non-dynamical variables.

Summarizing the previous analysis, the action of GR with the Holst modification can be written in a generic local Lorenz frame as follows

$$S=\int d^4x\left[{}^{(\gamma)}\widetilde{\pi}^i_a\partial_t{}^{(\gamma)}\widetilde{A}^a_i+\pi^a\partial_t\chi_a-\frac{1}{\sqrt{g}g^{tt}}H+\frac{g^{ti}}{g^{tt}}H_i+\eta^aG_a+\lambda_a\pi^a\right].$$

The set of Hamiltonian constraints reproduces a SU(2) gauge theory with the additional invariance under the action of 3-diffeomorphisms ($H_i=0$) and time re-parameterizations ($H=0$) - background-independence.

(FC, G. Montani, *PRL*, **102**, (2009), 091301.)

### Loop Quantum Gravity

As soon as a quantum description is addressed, any dependence from $\chi_a$ variables can be avoided (as for the lapse function and the shift vector).

Hence, *the standard quantization in terms of holonomies and fluxes of the SU(2) group works, even though no gauge fixing of the local Lorentz frame has been performed.*

Furthermore, since the Hilbert space is the same in any local Lorentz frame, *the discrete spatial structure proper of LQG is invariant under local Lorentz transformations.*

Summarizing the previous analysis, the action of GR with the Holst modification can be written in a generic local Lorenz frame as follows

$$S=\int d^4x\left[{}^{(\gamma)}\widetilde{\pi}^i_a\partial_t{}^{(\gamma)}\widetilde{A}^a_i+\pi^a\partial_t\chi_a-\frac{1}{\sqrt{g}g^{tt}}H+\frac{g^{ti}}{g^{tt}}H_i+\eta^aG_a+\lambda_a\pi^a\right].$$

The set of Hamiltonian constraints reproduces a SU(2) gauge theory with the additional invariance under the action of 3-diffeomorphisms ($H_i=0$) and time re-parameterizations ($H=0$) - background-independence.

(FC, G. Montani, *PRL*, **102**, (2009), 091301.)

### Loop Quantum Gravity

As soon as a quantum description is addressed, any dependence from $\chi_a$ variables can be avoided (as for the lapse function and the shift vector).

Hence, *the standard quantization in terms of holonomies and fluxes of the SU(2) group works, even though no gauge fixing of the local Lorentz frame has been performed.*

Furthermore, since the Hilbert space is the same in any local Lorentz frame, *the discrete spatial structure proper of LQG is invariant under local Lorentz transformations.*

# Matter fields

The same result concerning the non-dynamical role of $\chi_a$ have been obtained in presence of

- non-minimally coupled scalar fields;

(FC, G. Montani, *Phys. Rev. D*, **80**, (2009) 084045.)

- Immirzi scalar field: $\beta = 1/\gamma \Rightarrow \beta(x)$

  generalization of the second-class constraints solutions

$$\omega_a{}^b{}_i = {}^\pi\omega_a{}^b{}_i + \chi_a\omega^{0b} + \chi^b(\omega_a{}^0{}_i - {}^\pi D_i\chi_a) + {}^1\omega_a{}^b{}_i \qquad \pi^i_{ab} = 2\chi_{[a}\pi^i_{b]}$$

$${}^1\omega^{ab}_i = T^{[a}_c\left(-\frac{2(1+\chi^2)^2}{\chi^4+2\chi^2+2}\eta^{b]d} + \frac{2+\chi^2}{\chi^4+2\chi^2+2}\chi^{b]}\chi^d\right)\pi^c_i\pi^j_d\frac{\beta\partial_j\beta}{\beta^2+1}.$$

  At the end of the story, the kinematical sector coincides with the one of gravity minimally-coupled to a scalar field, while the dynamics differs significantly and the relaxation to a non-vanishing vacuum expectation value can be realized with natural assumptions.

(FC, G. Montani, *Phys. Rev. D*, **80**, (2009) 084040.)

- spinor fields with a non-minimal action

$$S_\psi = \int \sqrt{-g}\,\frac{i}{2}\left[\bar{\psi}\gamma^\mu(I + i\beta\gamma_5)D_\mu\psi - D_\mu(1 + i\beta\gamma_5)\bar{\psi}A\gamma^\mu\psi - 2im\bar{\psi}\psi\right]d^4x.$$

generalization of the second-class constraints solutions

$$\omega_a{}^b{}_i = {}^\pi\omega_a{}^b{}_i + \chi_a\omega^{0b} + \chi^b(\omega_a{}^0{}_i - {}^\pi D_i\chi_a) + {}^\psi\omega_a{}^b{}_i \qquad \pi^i_{ab} = 2\chi_{[a}\pi^i_{b]}$$

$${}^\psi\omega^{ab}_i = -\frac{1}{2}\frac{\gamma(\gamma-\beta)}{\gamma^2+1}\epsilon^{ab}{}_c\pi^c_i J^t_A - \frac{1}{2}\frac{\gamma(1+\beta\gamma)}{\gamma^2+1}\pi^c_i T_c{}^{-1[a}\epsilon^{b]fg}J_{fg},$$

$$(J^t_A = \sqrt{-g}\bar{\psi}\gamma^t\gamma_5\psi,\; J_{ab} = -\tfrac{1}{4}\sqrt{-g}\bar{\psi}\{\gamma^t,\Sigma_{ab}\}\psi.)$$

and redefinition of spinors $\psi = e^{i\chi^a\Sigma_{0a}}\psi^*$

SU(2) Gauss constraints still arise and $\chi_a$ non-dynamical.

$$\partial_i\widetilde{\pi}^i_a + \epsilon_{ab}{}^c\widetilde{A}^b_i\widetilde{\pi}^i_c = -\tfrac{1}{4}\sqrt{h}\bar{\psi}^*\gamma_5\gamma_a\psi^*, \qquad \pi^a = 0.$$

The super-momentum and the super-Hamiltonian differ from the analogous ones for a background-independent Yang-Mills gauge theories.

In presence of a proper non-minimal coupling

$$\beta = \gamma$$

a "pure" Yang-Mills interaction is predicted for the coupling between the geometry and the spinor

$$H_i = H_i^G + \tfrac{i}{2}\sqrt{h}(\bar{\psi}^*\gamma^0 A^{(A)}D_i\psi^* - {}^{(A)}D_i\bar{\psi}^* A\gamma^0\psi^*),$$

$$H = H^G - \tfrac{i}{2}\widetilde{\pi}^i_a\left(\bar{\psi}^*\gamma^a A^{(A)}D_i\psi^* - {}^{(A)}D_i\bar{\psi}^* A\gamma^a\psi^*\right) - \sqrt{h}\frac{3-\gamma^2}{4}\bar{\psi}^*\gamma_5\gamma_a\psi^*\bar{\psi}^*\gamma_5\gamma^a\psi^*.$$

$${}^{(A)}D_i\psi = \partial_i\psi - \gamma\widetilde{A}^a_i T_a\psi \qquad\qquad T_a = \epsilon_a{}^{bc}\Sigma_{bc}.$$

The Immirzi parameter finds a natural interpretation as the coupling constant of this SU(2) interaction.

# Conclusions

## Holst formulation

The Hamiltonian formulation of the Holst action for gravity has been analyzed and the emergence of SU(2) Gauss constraints has been outlined$\longrightarrow$ Loop Quantum Gravity in a generic local Lorentz frame.

Could this result be relevant for a path-integral formulation?

## In presence of matter fields

A Yang-Mills SU(2) gauge symmetry is still present and

- Immirzi field: interesting kinematical (Loop quantization) and dynamical implications.
- spinor: intriguing correspondence with a background-independent SU(2) gauge theory$\longrightarrow$ Path integral formulation is promising (4-fermions terms).

# Conclusions

## Holst formulation

The Hamiltonian formulation of the Holst action for gravity has been analyzed and the emergence of SU(2) Gauss constraints has been outlined$\longrightarrow$ Loop Quantum Gravity in a generic local Lorentz frame.

Could this result be relevant for a path-integral formulation?

## In presence of matter fields

A Yang-Mills SU(2) gauge symmetry is still present and

- Immirzi field: interesting kinematical (Loop quantization) and dynamical implications.
- spinor: intriguing correspondence with a background-independent SU(2) gauge theory$\longrightarrow$ Path integral formulation is promising (4-fermions terms).