Conceptual issues in numerical relativity

Jörg Frauendiener

Department of Mathematics and Statistics, University of Otago, New Zealand
and
Centre of Mathematics for Applications, Oslo, Norway

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1. Why Computational Gravity?

2. Some examples
    - Bartnik-McKinnon solutions
    - Critical collapse
    - The BKL conjecture
    - Gravitational waves and binary mergers

3. Conceptual issues

4. Visualisations

Why Computational Gravity?

Einstein 1915

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final form of the field equations for gravity

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final form of the field equations for gravity

That's it?

Why Computational Gravity?

Einstein 1915
final form of the field equations for gravity

Of course not!

Maxwell theory

Example (Maxwell's theory of electrodynamics)

Sommerfeld's book on Electrodynamics (1949)

Part 1 Foundations and basic notions of Maxwell's electrodynamics

Part 2 Derivation of the phenomena from Maxwell's equations

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Deductive procedure

- specify different scenarios (matter model, symmetry, etc)

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Only the solutions give information about content and value of a theory

Timescales for Maxwell's and Einstein's theory

	Maxwell	**Einstein**
theory in final form	1864	1915
waves predicted	1864	1917
clarified	—	1962
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Einstein's theory is conceptually and technically much more challenging

Role for computations

provide **solutions** for

- explorations
  - phase-space of GR is large
  - 'see what happens'
- theoretical purposes
  - formulate/verify/refute conjectures
- experiments
  - very (?) specific situations (cp. exact solutions)
  - predictions (wave templates)

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Bartnik-McKinnon solutions

Bartnik-McKinnon, 1988

- “strong numerical evidence” for a family of particle-like solutions of $SU(2)$-EYM equations
- later also black hole solutions found
- in contrast to black holes have no hair beliefs
- question of stability properties for solutions
- sparked a vast amount of mathematical investigations
    - rigorous existence proofs (for arbitrary gauge groups)
    - rigorous stability results (for arbitrary gauge groups)
    - generalisations to $\Lambda \neq 0$

Bartnik-McKinnon solutions

Concentration on energy density

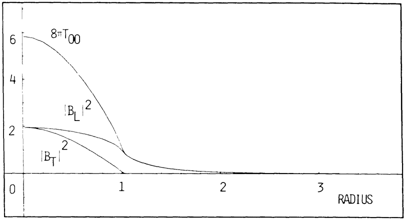

FIG. 2. Energy density in the interior, $k=3$.

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Critical collapse

Choptuik, 1993

- study of spherically symmetric scalar field collapse (arbitrarily small mass?)
- discovered critical behaviour of solutions near dispersion/collapse boundary
- bh mass scales as a power-law

$$M \propto (p - p_*)^\gamma$$

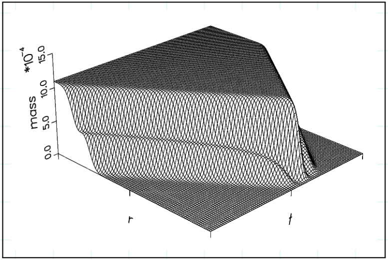
- critical exponent γ independent of ic
- approach to discrete self-similar solution (DSS, echoing) in strong curvature region close to the collapse,independent of ic

$\therefore$ universal behaviour (within one model)

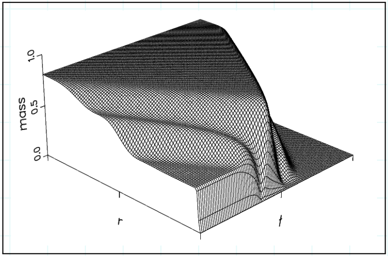
- importance of very high spatial resolution

Critical collapse

Two evolutions



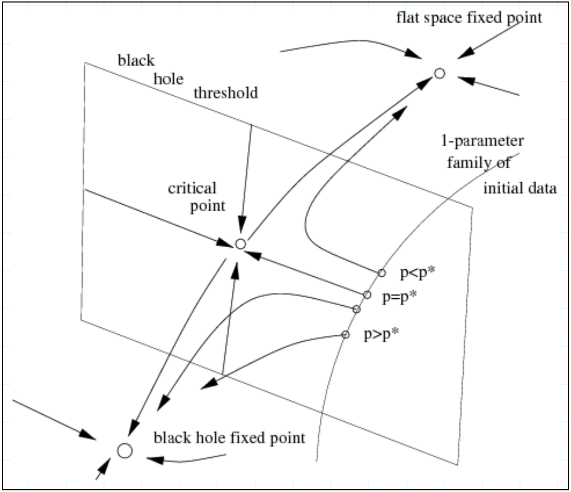
dispersion to infinity



black hole

Critical collapse

Phase space



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Critical collapse

- Critical phenomena have since then been observed in many other relativistic systems
- many different numerical implementations used
- connection with phase transitions in statistical physics
- rigorous theoretical understanding is still lacking
- heuristic explanations use renormalisation group methods
- dynamical systems approach
- so far no exact solution showing DSS behaviour has been found

The BKL conjecture

- Singularities develop under very general circumstances (Penrose et al)
- no information about the nature of these singularities
- heuristic study by Belinskii, Khalatnikov and Lifshitz

BKL conjecture

The approach to a generic singularity becomes local and oscillatory (Mixmaster)

- Berger, Moncrief (1993) use numerical methods to investigate:
  - BKL seems to be true: spatial derivatives become unimportant, oscillatory behaviour except for isolated points
  - but not conclusive: high symmetry (Gowdy), lack of resolution, 'spiky features'
- Uggla et al. (2003) devise set of scale-invariant variables, used to formulate the conjecture precisely
- calculations by Garfinkle in the general case support the BKL conjecture but resolution is still too low to be conclusive

The BKL conjecture

The spikes

- higher resolution confirms the unexpected presence of the spikes
- Rendall, Weaver construct explicit solutions with spikes from solutions without spikes (Bäcklund type trafo)
- they find true (geometric) and false (coordinate) spikes
- leads to rigorous proof of existence of Gowdy space-times with spikes
- spikes have been found in more general cosmological space-times with an intriguing dynamics

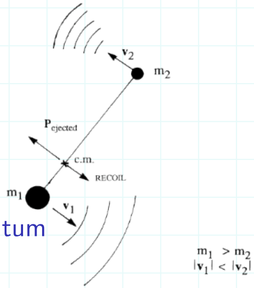
Gravitational waves and binary mergers

The holy grail

- first attempts in the 1950's by L. Smarr and others
- many hundreds of man-years
- slow progress due to lack of understanding on several levels:
  - mathematical well-posedness, hyperbolicity
  - constraint propagation and damping
  - non-linear self-interaction
  - efficiency, accuracy and stability
- break-through Pretorius (2005)
- now several groups use different codes and methods produce very similar results
- field has developed almost into an engineering science

Gravitational waves and binary mergers

The kick

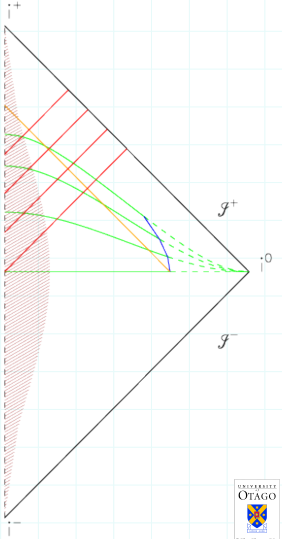


- gw from merging binaries carries away energy-momentum
- the system gets a kick → recoil velocity
- depends critically on momenta and spins of the partners
- computed up to $3.300 - 4.000\text{km/s}$
- observable consequences in astrophysics
  bh can be ejected from their host system
- possible candidates have been identified
- probably the single most important prediction of CG in this area

Conceptual issues

Gravitational radiation at null-infinity $\mathscr{I}$

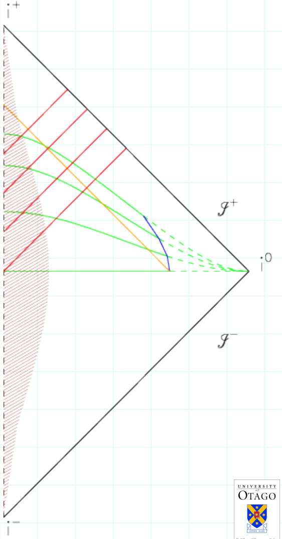
- evolution proceeds on space-like hypersurfaces
- radiation travels on null-hypersurfaces
- grid boundary is time-like hypersurface
- influences radiation along a null-hypersurface
- ultimately gw hit the grid boundary
  → uncontrolled interaction
- needs a 'transparent' boundary condition



Conceptual issues

What boundary condition?

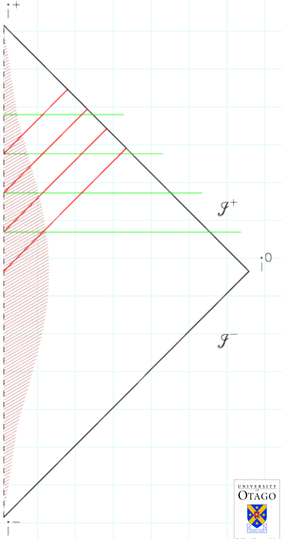
- numerical stability
- mathematical well-posedness of IBVP
- physically relevant, transparent
- outgoing radiation at boundary is observer dependent
- currently:
  stay away from the influence of the boundary
  damp at the boundary
- accuracy depends on the boundary treatment



Conceptual issues

Conformal space-time

- Penrose compactification (1963)
  attach a regular boundary to space-time
- evolve with Friedrich's CFE
- foliate with space-like hypersurfaces intersecting $\mathscr{I}$
- $\mathscr{I}$ is characteristic → no bc necessary
- pick up the unique wave signal on $\mathscr{I}$
- conceptually less approximations:
  use as 'reference'



Visualisations

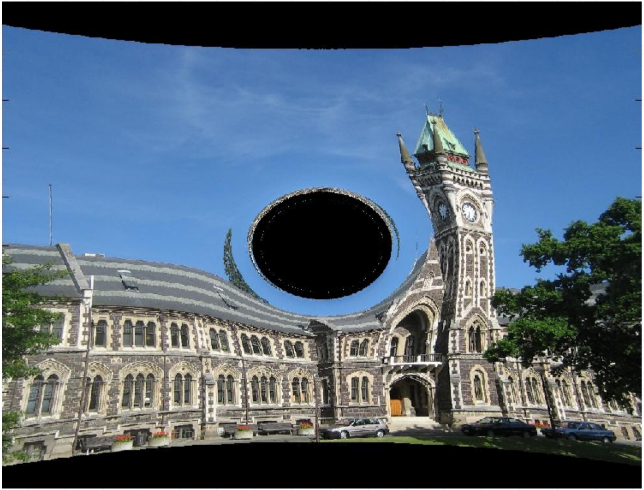
Special relativity

`eiffel.mpg`

(Th Müller, Uni Tübingen)

Visualisations

General relativity



(TeYu Chyou, Uni Otago)

Visualisations

General relativity

`earth.mov`

(Th Müller, Uni Tübingen)