The Kerr-Schild ansatz revised

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Kerr-Schild metrics

Kerr-Schild metrics have the form [1, 2]

$$\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \equiv (\eta_{\alpha\beta} - 2Hk_\alpha k_\beta)\mathrm{d}x^\alpha \mathrm{d}x^\beta \ ,$$

where $\eta_{\alpha\beta}$ is the metric for Minkowski space and k_α is a null vector

$$\eta_{\alpha\beta}k^\alpha k^\beta = g_{\alpha\beta}k^\alpha k^\beta = 0, \qquad k^\alpha = \eta^{\alpha\beta}k_\beta = g^{\alpha\beta}k_\beta \ .$$

The inverse metric is also linear in H

$$g^{\alpha\beta} = \eta^{\alpha\beta} + 2Hk^\alpha k^\beta \ ,$$

and so the determinant of the metric is independent of H

$$(\eta_{\alpha\gamma} - 2Hk_\alpha k_\gamma)(\eta^{\gamma\beta} + 2Hk^\gamma k^\beta) = \delta_\alpha^\beta \quad \longrightarrow \quad |g_{\alpha\beta}| = |\eta_{\alpha\beta}| \ .$$

Kerr solution

Within this class of general metrics the Kerr solution was obtained in 1963 by a systematic study of algebraically special vacuum solutions [3]. If $(x^0 = t, x^1 = x, x^2 = y, x^3 = z)$ are the standard Cartesian coordinates for Minkowski spacetime with $\eta_{\alpha\beta} = \mathrm{diag}[-1,1,1,1]$, then for Kerr metric we have

$$-k_\alpha \mathrm{d}x^\alpha = \mathrm{d}t + \frac{(rx+ay)\mathrm{d}x + (ry-ax)\mathrm{d}y}{r^2+a^2} + \frac{z}{r}\mathrm{d}z \ , \tag{5}$$

where r and H are defined implicitly by

$$\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1 \ , \qquad H = -\frac{\mathcal{M}r^3}{r^4+a^2z^2} \ . \tag{6}$$

Kerr solution is asymptotically flat and the constants $\mathcal{M}$ and a are the total mass and specific angular momentum for a localized source. They both have

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Modified ansatz

1) consider Kerr-Schild metrics as *exact linear perturbations* of Minkowski space
2) solve Einstein's field equations order by order in powers of H

Let ϵ be an arbitrary constant parameter, eventually to be set equal to 1, so that the Kerr-Schild metric (1) reads

$$g_{\alpha\beta} = \eta_{\alpha\beta} - 2\epsilon H k_\alpha k_\beta \ , \tag{7}$$

with inverse

$$g^{\alpha\beta} = \eta^{\alpha\beta} + 2\epsilon H k^\alpha k^\beta \ , \tag{8}$$

and suppose that coordinates are chosen so that the components $\eta_{\alpha\beta}$ are constants, but not necessarily of the form $\eta_{\alpha\beta} = \text{diag}[-1, 1, 1, 1]$. The connection

The ϵ-expansion

Connection:

$$\Gamma^{\gamma}{}_{\alpha\beta} = \epsilon \underset{1}{\Gamma}{}^{\gamma}{}_{\alpha\beta} + \epsilon^2 \underset{2}{\Gamma}{}^{\gamma}{}_{\alpha\beta} \ ,$$

where

$$\underset{1}{\Gamma}{}^{\gamma}{}_{\alpha\beta} = -(Hk_\alpha k^\gamma)_{,\beta} - (Hk_\beta k^\gamma)_{,\alpha} + (Hk_\alpha k_\beta)_{,\lambda}\eta^{\lambda\gamma} \ ,$$
$$\underset{2}{\Gamma}{}^{\gamma}{}_{\alpha\beta} = 2H[H(\dot{k}_\alpha k_\beta + \dot{k}_\beta k_\alpha) + \dot{H} k_\alpha k_\beta]k^\gamma \equiv 2Hk^\gamma (Hk_\alpha k^\beta)^{\cdot} \ ,$$

a "dot" denoting differentiation in the $\boldsymbol{k}$ direction, i.e. $\dot{f} = \boldsymbol{k}(f) = f_{,\alpha}k^\alpha$. Note that only the indices of $\boldsymbol{k}$ can be raised and lowered with the Minkowski metric. Hereafter we will use an "index" 0 to denote contraction with $\boldsymbol{k}$, i.e.

$$\begin{aligned}
\Gamma^{0}{}_{\alpha\beta} &= \Gamma^{\gamma}{}_{\alpha\beta}k_\gamma = \epsilon(Hk_\alpha k_\beta)^{\cdot} \ , \\
\Gamma^{\gamma}{}_{\alpha 0} &= \Gamma^{\gamma}{}_{\alpha\beta}k^\beta = -\epsilon(Hk_\alpha k^\gamma)^{\cdot} \ , \\
\Gamma^{\gamma}{}_{00} &= \Gamma^{\gamma}{}_{\alpha\beta}k^\alpha k^\beta = 0 \ , \\
\Gamma^{0}{}_{\alpha 0} &= \Gamma^{\gamma}{}_{\alpha\beta}k^\beta k_\gamma = 0 \ .
\end{aligned} \tag{11}$$

Ricci tensor:

$$R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta} = \Gamma^{\gamma}{}_{\alpha\beta,\gamma} - \Gamma^{\gamma}{}_{\alpha\delta}\Gamma^{\delta}{}_{\beta\gamma}$$

$$= \epsilon \underset{1}{R}{}_{\alpha\beta} + \epsilon^2 \underset{2}{R}{}_{\alpha\beta} + \epsilon^3 \underset{3}{R}{}_{\alpha\beta} + \epsilon^4 \underset{4}{R}{}_{\alpha\beta}$$

$$\underset{4}{R}{}_{\alpha\beta} = - \underset{2}{\Gamma}{}^{\rho}{}_{\alpha\sigma} \underset{2}{\Gamma}{}^{\sigma}{}_{\beta\rho} = 0 \ ,$$

$$\underset{3}{R}{}_{\alpha\beta} = - \underset{1}{\Gamma}{}^{\rho}{}_{\alpha\sigma} \underset{2}{\Gamma}{}^{\sigma}{}_{\beta\rho} - \underset{2}{\Gamma}{}^{\rho}{}_{\alpha\sigma} \underset{1}{\Gamma}{}^{\sigma}{}_{\beta\rho} = 4H^3 ||\dot{\boldsymbol{k}}||^2 k_\alpha k_\beta$$

→ $||\dot{\boldsymbol{k}}|| = 0$ and so $\dot{\boldsymbol{k}}$ is a null-vector orthogonal

to another null-vector, $\boldsymbol{k}$. Hence $\dot{\boldsymbol{k}}$ must be parallel to $\boldsymbol{k}$ and therefore $\boldsymbol{k}$ is a geodesic vector.

$$\underset{2}{R}_{\alpha\beta} = \underset{2}{\Gamma}^{\rho}{}_{\alpha\beta,\rho} - \underset{1}{\Gamma}^{\rho}{}_{\alpha\sigma}\,\underset{1}{\Gamma}^{\sigma}{}_{\beta\rho}$$
$$= 2H\left[(Hk_\alpha k_\beta)^{\cdot\cdot} + k^{\sigma}{}_{,\sigma}(Hk_\alpha k_\beta)^{\cdot} - H\dot{k}_\alpha \dot{k}_\beta\right]$$
$$- H^2\Phi k_\alpha k_\beta - 2Hk_{(\alpha}\psi_{\beta)}\ ,$$

$$\underset{1}{R}_{\alpha\beta} = \underset{1}{\Gamma}^{\gamma}{}_{\alpha\beta,\gamma}$$
$$= Ak_\alpha k_\beta + 2k_{(\alpha}B_{\beta)} + X_{\alpha\beta}\ ,$$

where

$$\Phi = 4\eta^{\gamma\lambda}\eta^{\delta\mu}k_{[\lambda,\delta]}k_{[\mu,\gamma]}\ , \qquad \psi_\alpha = 2\dot{k}^{\gamma}(Hk_\alpha)_{,\gamma}$$

$$A = \eta^{\lambda\gamma}H_{,\lambda\gamma}\ ,$$
$$B_\beta = -(Hk^{\gamma})_{,\gamma\beta} + \frac{1}{H}\eta^{\lambda\gamma}(H^2k_{\beta,\gamma})_{,\lambda}\ ,$$
$$X_{\alpha\beta} = -2H\left[(k_{(\alpha,\beta)}k^{\gamma})_{,\gamma} + k_{(\alpha,|\gamma|}k^{\gamma}{}_{,\beta)} - \eta^{\lambda\gamma}k_{\alpha,\gamma}k_{\beta,\lambda}\right]$$
$$- 2k^{\gamma}\left[H_{,(\alpha}k_{\beta),\gamma} + H_{,\gamma}k_{(\alpha,\beta)}\right]$$
$$= -2H\left[\dot{k}_{(\alpha,\beta)} + k^{\gamma}{}_{,\gamma}k_{(\alpha,\beta)} - \eta^{\lambda\gamma}k_{\alpha,\gamma}k_{\beta,\lambda}\right]$$
$$- 2\dot{H}k_{(\alpha,\beta)} - 2H_{,(\alpha}\dot{k}_{\beta)}\ .$$

Kinematical properties of the congruence k

Taking the covariant derivative of $\boldsymbol{k}$ gives

$$\nabla_\alpha k_\beta = k_{\beta,\alpha} - \epsilon (H k_\alpha k_\beta)^{\cdot} \quad ,$$

so that its 4-acceleration is simply

$$a(k)_\beta = k^\mu \nabla_\mu k_\beta = \dot{k}_\beta \ .$$

The other optical scalars of interest are the expansion

$$\theta = \frac{1}{2} k^\alpha{}_{;\alpha} = \frac{1}{2} \eta^{\alpha\beta} k_{\beta,\alpha} = \frac{1}{2} k^\alpha{}_{,\alpha} \ ,$$

the vorticity

$$\omega^2 = \frac{1}{2} k_{[\alpha;\beta]} k^{\alpha;\beta} = \frac{1}{2} k_{[\beta,\alpha]} \left(\eta^{\alpha\mu} \eta^{\beta\nu} k_{\mu,\nu} - 2\epsilon H \dot{k}^\alpha k^\beta \right) \ ,$$

and the shear, implicitly defined by

$$\theta^2 + |\sigma|^2 = \frac{1}{2} k_{(\alpha;\beta)} k^{\alpha;\beta} = \frac{1}{2} k_{(\beta,\alpha)} \eta^{\alpha\mu} \eta^{\beta\nu} k_{\mu,\nu} - \frac{1}{2} \epsilon H ||\dot{\boldsymbol{k}}||^2 \ .$$

First result: k be geodesic

The third order field equations (17) imply that $\boldsymbol{k}$ be geodesic. Then it can be normalized so that $\dot{\boldsymbol{k}} = 0$. The optical scalars (25) and (26) thus become

$$\begin{aligned} \omega^2 &= \frac{1}{2}\eta^{\alpha\mu}\eta^{\beta\nu}k_{[\beta,\alpha]}k_{\mu,\nu} \ , \\ \theta^2 + |\sigma|^2 &= \frac{1}{2}\eta^{\alpha\mu}\eta^{\beta\nu}k_{(\beta,\alpha)}k_{\mu,\nu} \ . \end{aligned} \tag{27}$$

The second order Ricci tensor (18) simplifies to

$$\underset{2}{R}{}_{\alpha\beta} = 2H\mathcal{D}k_\alpha k_\beta \ , \qquad \mathcal{D} = \ddot{H} + 2\theta\dot{H} + 4H\omega^2 \ ,$$

leading to the condition $\mathcal{D} = 0$, which gives the following equation for H

$$0 = \ddot{H} + 2\theta\dot{H} + 4H\omega^2 \ .$$

Finally, the first order Ricci tensor (20)–(21) becomes

$$\begin{aligned} \underset{1}{R}{}_{\alpha\beta} = {} & \eta^{\lambda\gamma}H_{,\lambda\gamma}k_\alpha k_\beta + 2k_{(\alpha}B_{\beta)} \\ & - 2\left[(\dot{H} + 2\theta H)k_{(\alpha,\beta)} - \eta^{\lambda\gamma}Hk_{\alpha,\gamma}k_{\beta,\lambda}\right] \ , \end{aligned}$$

with

$$B_\beta = -(\dot{H} + 2\theta H)_{,\beta} + \eta^{\lambda\gamma}(2H_{,\lambda}k_{\beta,\gamma} + Hk_{\beta,\gamma\lambda}) \ .$$

Simplified tetrad procedure

Following [5, 6] introduce the set of null coordinates in Minkowski space $(u, v, \zeta, \bar{\zeta}$ which are related to the standard Cartesian coordinates (t, x, y, z) by

$$u = \frac{1}{\sqrt{2}}(t - z) \ , \qquad v = \frac{1}{\sqrt{2}}(t + z) \ ,$$
$$\zeta = \frac{1}{\sqrt{2}}(x + iy) \ , \qquad \bar{\zeta} = \frac{1}{\sqrt{2}}(x - iy) \ . \tag{33}$$

The metric (7) becomes

$$\mathrm{d}s^2 = 2(\mathrm{d}\zeta \mathrm{d}\bar{\zeta} - \mathrm{d}u\mathrm{d}v) - 2\epsilon H k_\alpha k_\beta \mathrm{d}x^\alpha \mathrm{d}x^\beta \ . \tag{34}$$

A general field of real null directions in Minkowski space is given by

$$k = -[\mathrm{d}u + Y\bar{Y}\mathrm{d}v + \bar{Y}\mathrm{d}\zeta + Y\mathrm{d}\bar{\zeta}] \ , \qquad \boldsymbol{k} = Y\bar{Y}\partial_u + \partial_v - Y\partial_\zeta - \bar{Y}\partial_{\bar{\zeta}} \ , \tag{35}$$

where Y is an arbitrary complex function of coordinates. In fact the indepen-

We introduce the following frame

$$\omega^1 = \mathrm{d}\zeta + Y\mathrm{d}v\ , \quad \omega^2 = \mathrm{d}\bar{\zeta} + \bar{Y}\mathrm{d}v\ , \quad \omega^3 = -k\ , \quad \omega^4 = \mathrm{d}v + \epsilon H\omega^3\ , \tag{36}$$

so that

$$\mathrm{d}s^2 = 2\omega^1\omega^2 - 2\omega^3\omega^4\ . \tag{37}$$

The dual frame is

$$\boldsymbol{e}_1 = \partial_\zeta - \bar{Y}\partial_u\ , \quad \boldsymbol{e}_2 = \partial_{\bar{\zeta}} - Y\partial_u\ , \quad \boldsymbol{e}_3 = \partial_u - \epsilon H\boldsymbol{k}\ , \quad \boldsymbol{e}_4 = \boldsymbol{k}\ . \tag{38}$$

The connection coefficients are given by

$$\Gamma_{cab} = -e_c{}^\mu e_{a\,\mu;\nu} e_b{}^\nu\ . \tag{39}$$

The derivative of $\boldsymbol{k}$ is quite simple

$$k_{\mu,\nu} = k_{\mu,\bar{Y}}\bar{Y}_{,\nu} + k_{\mu,\bar{Y}}\bar{Y}_{,\nu} = -\omega^1_\mu\bar{Y}_{,\nu} - \omega^2_\mu Y_{,\nu}\ .$$

Next introduce the following standard notation for the directional derivatives along the frame vectors

$$\begin{aligned} D \equiv \nabla_{\boldsymbol{k}} &= \partial_v + Y\bar{Y}\partial_u - Y\partial_\zeta - \bar{Y}\partial_{\bar{\zeta}} \ , \\ \Delta \equiv \nabla_{\boldsymbol{e}_3} &= \partial_u - \epsilon H D \ , \\ \delta \equiv \nabla_{\boldsymbol{e}_1} &= \partial_\zeta - \bar{Y}\partial_u \ . \end{aligned} \tag{42}$$

The geodesic curvature κ, complex expansion ρ and shear σ of the null congruence $\boldsymbol{k}$ are given by

$$\begin{aligned} \kappa &\equiv -\Gamma_{414} = -k^\alpha D e_{1\alpha} = D\bar{Y} \ , \\ \rho &\equiv -\Gamma_{412} = -k^\alpha \bar{\delta} e_{1\alpha} = \bar{\delta}\bar{Y} \ , \\ \sigma &\equiv -\Gamma_{411} = -k^\alpha \delta e_{1\alpha} = \delta\bar{Y} \ , \end{aligned} \tag{43}$$

respectively. It is also useful to introduce the quantity

$$\tau \equiv -\Gamma_{413} = -k^\alpha \Delta e_{1\alpha} = \partial_u \bar{Y} \ . \tag{44}$$

the principal null vector $\boldsymbol{k}$ is geodesic, then $\kappa = 0$, i.e.

$$0 = D\bar{Y} = \bar{Y}_{,v} + Y\bar{Y}\bar{Y}_{,u} - Y\bar{Y}_{,\zeta} - \bar{Y}\bar{Y}_{,\bar{\zeta}} \ .$$

Completion of the solution

In terms of the connection coefficients previously introduced the optical scalars write as

$$\theta = -\frac{1}{2}(\rho + \bar{\rho}) \ , \qquad \omega^2 = -\frac{1}{4}(\rho - \bar{\rho})^2 \ , \tag{51}$$

so that the single equation (29) coming from the vanishing of second order Ricci tensor reads

$$0 = \ddot{H} - (\rho + \bar{\rho})\dot{H} - (\rho - \bar{\rho})^2 H \ . \tag{52}$$

The nonvanishing relevant frame components of the first order Ricci tensor (30) are given by

$$\underset{1}{R}_{11} = \ 2\sigma[\dot{H} - (\bar{\rho} - \rho)H] \ , \tag{53a}$$

$$\underset{1}{R}_{12} = (\rho + \bar{\rho})\dot{H} - (\rho^2 + \bar{\rho}^2 - 2\sigma\bar{\sigma})H \ , \tag{53b}$$

$$\underset{1}{R}_{13} = \delta\dot{H} + (\rho - \bar{\rho})\delta H + 2\sigma\bar{\delta}H - \tau\dot{H} - (\delta\bar{\rho} + 2\bar{\tau}\sigma + 2\tau\rho - \bar{\delta}\sigma)H \ , \tag{53c}$$

$$\underset{1}{R}_{33} = 2\left[\delta\bar{\delta}H - (\rho_{,u} + \bar{\rho}_{,u})H - \tau\bar{\delta}H - \bar{\tau}\delta H - \rho H_{,u}\right] \ , \tag{53d}$$

$$\underset{1}{R}_{34} = \ddot{H} - (\rho + \bar{\rho})\dot{H} - (\rho - \bar{\rho})^2 H \ , \tag{53e}$$

since $\underset{1}{R}_{22}$ and $\underset{1}{R}_{23}$ are c.c. of $\underset{1}{R}_{11}$ and $\underset{1}{R}_{13}$ respectively. The identities

Equation (53a) implies $\sigma = 0$, i.e. the congruence $\boldsymbol{k}$ must be shearfree. The remaining first order equations thus simplify as

$$0 = (\rho + \bar{\rho})\dot{H} - (\rho^2 + \bar{\rho}^2)H \ , \tag{56a}$$

$$0 = \delta\dot{H} + (\rho - \bar{\rho})\delta H - \tau\dot{H} - (\delta\bar{\rho} + 2\tau\rho)H \ , \tag{56b}$$

$$0 = \delta\bar{\delta}H - (\rho_{,u} + \bar{\rho}_{,u})H - \tau\bar{\delta}H - \bar{\tau}\delta H - \rho H_{,u} \ , \tag{56c}$$

the solution is

$$H = \frac{1}{2}M(\rho + \bar{\rho})$$

$$P = (M/m)^{-1/3}$$

$$P = pY\bar{Y} + qY + \bar{q}\bar{Y} + c \ ,$$

where p and c are real constants and q is a complex constant.

Finally, taking the exterior derivative of Y gives

$$\begin{aligned} \mathrm{d}Y &= \delta Y \omega^1 + Y_u \omega^3 = P^{-1}\bar{\rho}[P\omega^1 - P_{,\bar{Y}}\omega^3] \\ &= P^{-1}\bar{\rho}[(qY+c)(\mathrm{d}\zeta + Y\mathrm{d}v) - (pY+\bar{q})(\mathrm{d}u + Y\mathrm{d}\bar{\zeta})] \ , \end{aligned} \tag{72}$$

whose general solution is

$$0 = F \equiv \phi(Y) + (qY+c)(\zeta + Yv) - (pY+\bar{q})(u + Y\bar{\zeta}) \ , \tag{73}$$

according to Eq. (50), with ϕ an arbitrary analytic function of the complex variable Y. In fact, differentiating Eq. (73) leads to

Summarizing, the solution is given by

$$\mathrm{d}s^2 = 2(\mathrm{d}\zeta\mathrm{d}\bar{\zeta} - \mathrm{d}u\mathrm{d}v) - \frac{m}{P^3}(\rho+\bar{\rho})[\mathrm{d}u + Y\bar{Y}\mathrm{d}v + \bar{Y}\mathrm{d}\zeta + Y\mathrm{d}\bar{\zeta}]^2 \ ,$$

with

$$P = pY\bar{Y} + qY + \bar{q}\bar{Y} + c \ , \qquad \bar{\rho} = PF_{,Y}{}^{-1} \ .$$

Kerr solution:

$$P = (1+Y\bar{Y})/\sqrt{2}, \qquad \phi \ = \ -iaY, \qquad m = \mathcal{M}$$

Conclusions

- We have presented an alternative derivation of Kerr solution by treating Kerr-Schild metrics as *exact linear perturbations* of Minkowski spacetime.

  In fact they have been introduced as a linear superposition of the flat spacetime metric and a squared null vector field ***k*** multiplied by a scalar function *H*.

- In the case of Kerr solution the vector ***k*** is geodesic and shearfree and it is independent of the mass parameter *M*, which enters instead the definition of *H* linearly.

  This linearity property allows one to solve the field equations order by order in powers of *H* in complete generality, i.e. without any assumption on the null congruence ***k***.

  The Ricci tensor turns out to consist of three different contributions.

  Third order equations all imply that ***k*** must be geodesic; it must be also shearfree as a consequence of first order equations, whereas the solution for *H* comes from second order equations too.

- Generalization to Kerr-Newman: ***k*** depends only on the rotation parameter *a* and not on the mass *M* or charge *Q*. Furthermore, the electromagnetic field is linear in *Q* and the metric is linear in *M* and Q^2 since the function *H* is obtained simply by replacing

  $M \rightarrow M - Q^2/(2r)$.