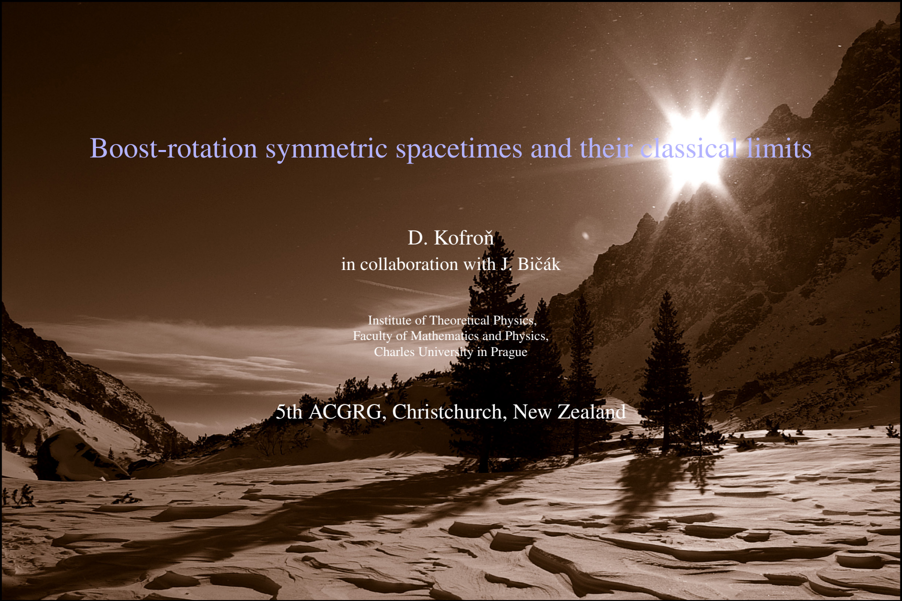

Boost-rotation symmetric spacetimes and their classical limits

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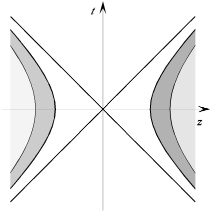
Contents

Vacuum boost-rotation symmetric spacetimes

$$ds^2 = \frac{e^{\mu}\left(z\mathrm{d}t - t\mathrm{d}z\right)^2 - e^{\nu}\left(z\mathrm{d}z - t\mathrm{d}t\right)^2}{z^2 - t^2} - e^{\nu}\mathrm{d}r^2 - e^{-\mu}r^2\mathrm{d}\phi^2$$



worldlines of the sources

- global coordinates
- asymptotical flatness (for spatially bounded sources)
- spacelike axial Killing vector and boost Killing vector
- hypersurfaces defined by null norm of the boost Killing vector divide the spacetime into four quadrants
- above the roof – locally cyllindrical waves
- below the roof – locally Weyl spacetimes
- the Einstein field equations reduce to linear wave equation

  $$\Box\mu = 0$$

  in the auxiliary flat spacetime
- solutions of algebraic type I, in general; radiative
- the only explicite solutions describing accelerated sources known

The interpretation is based upon special-relativistic limit in which the worldlines of sources are orbits of boost Killing vector. These spacetimes are considered to describe (nontrivially) moving massive particles.

Bičák, J., Schmidt, B.: Asymptotically flat radiative space-times with boost-rotation symmetry: The general structure. Phys. Rev. D **40**, 1827–1853 (1989)

Electrovacuum rotating boost-rotation symmetric spacetimes; the rotating charged C-metric

- the electrovacuum rotating solutions are generalization of the previous case
- both of these generalizations lead to non-linear Ernst equations (which reduce to linear equation only in the non-rotating vacuum case)
- difficult to solve
- generating techniques lead to singularities or breaking the asymptotical flatness

But, almost by a miracle, there exist algebraically special (of Petrov type D) charged rotating solution called the C-metric

$$
\begin{aligned}
\mathrm{d}s^2 = \frac{1}{A^2(x-y)^2}\Bigg\{ & \frac{\mathcal{G}(y)}{1+(aAxy)^2}\left[\left(1+a^2A^2x^2\right)K\mathrm{d}t + aA\left(1-x^2\right)K\mathrm{d}\phi\right]^2 - \frac{1+(aAxy)^2}{\mathcal{G}(y)}\,\mathrm{d}y^2 \\
& + \frac{1+(aAxy)^2}{\mathcal{G}(x)}\,\mathrm{d}x^2 + \frac{\mathcal{G}(x)}{1+(aAxy)^2}\left[\left(1+a^2A^2y^2\right)K\mathrm{d}\phi + aA\left(y^2-1\right)K\mathrm{d}t\right]^2\Bigg\},
\end{aligned}
$$

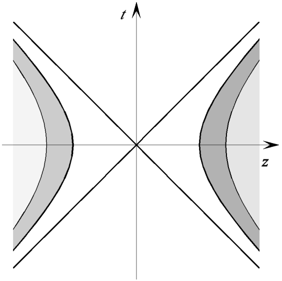
where

$$
\mathcal{G}(\xi) = \left(1-\xi^2\right)\left(1+r_+A\xi\right)\left(1+r_-A\xi\right),
$$

with

$$
r_\pm = Gm \pm \sqrt{G^2m^2 - a^2 - Gq^2}\,.
$$

Conical singularities



$r = 0$

Conical singularities are important properties of these solutions.

- they extend along the symmetry axis
- they are the sources of acceleration (or, more precisely, the remnants of the fact, that the physical reason of acceleration is not included)
- they are interpreted as strings with inner tension proportional to the acceleration

These singularities can be removed (but then the asymptotical flatness is lost)

- by adding of the external gravitational field
- by adding of the external homogeneous electric field
- by fine tunning of the parameters of special Bonnor-Swaminarayan solution

Then we have the exact singularity free solution (expect the sources themselves) of Einstein field equations describing accelerated sources of gravity and electromagnetic field.

Immersing the C-metric in an external electric field

Immersing the charged rotating C-metric in an external electric field is one way how to eliminate conical singularities. This is an old idea by Ernst.
Technically, this can be done by the so-called Harrison transformation

$$\hat{\mathcal{E}} = \mathcal{E}\Lambda^{-1}\,, \qquad \hat{\Phi} = (\Phi + \gamma\mathcal{E})\,\Lambda^{-1}\,, \qquad \Lambda = 1 - 2\gamma\Phi - \bar{\gamma}\gamma\mathcal{E}\,. \tag{1}$$

The reconstructed metric (in nonrotating case) reads

$$\mathrm{d}s^2 = \frac{1}{A^2(x-y)^2}\left[\bar{\Lambda}\Lambda\left(\mathcal{G}(y)\,\mathrm{d}t^2 - \frac{\mathrm{d}y^2}{\mathcal{G}(y)} + \frac{\mathrm{d}x^2}{\mathcal{G}(x)}\right) + \frac{1}{\bar{\Lambda}\Lambda}\,\mathcal{G}(x)\,\mathrm{d}\phi^2\right].$$

The axis is regular if

$$\lim \frac{F^{,a}F_{,a}}{4F} = 1\,, \tag{2}$$

where $F = \xi^a\xi_a$.

Immersing the C-metric in an external electric field

Employing the new form of the C-metric (by Hong and Teo) we can express the reuglarity condition for the "generalized" C-metric as which in the case of "generalized" C-metric yields

$$\lim_{x\to\pm 1} \hat{\delta} = \frac{K^2\left(1+a^2A^2+G\left(q^2A^2\pm 2mA\right)\right)^2}{\left[\left(1\pm\frac{1}{2}GKEq\right)^4+G^4K^4E^4a^2m^2\right]^2}\,, \tag{3}$$

The system of two equations $\lim_{x\to 1}\hat{\delta} = \lim_{x\to -1}\hat{\delta} = 1$ can be solved exacly

$$K = \frac{1}{1+a^2A^2} - \frac{q^2A^2}{(1+a^2A^2)^2}\,G + \frac{1}{2}\frac{\left(2q^2A^4+3E^2\right)q^2}{(1+a^2A^2)^3}\,G^2 + O(G^3)\,, \tag{4}$$

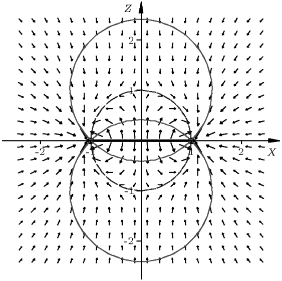
$$m = \frac{qE}{A}\left[1+\frac{1}{4}\frac{E^2q^2}{(1+a^2A^2)}\,G^2 - \frac{1}{2}\frac{E^2q^4A^2}{(1+a^2A^2)^3}\,G^3 + O(G^4)\right]. \tag{5}$$

Where q and m are just "meaningless" parameters. The net physical charge is $Q = Kq$ and so

$$\frac{m}{1+a^2A^2} = \frac{EQ}{A}\left[1+\frac{1}{4}E^2Q^2G^2 - \frac{1}{2}\left(1+a^2A^2\right)E^2Q^4A^2G^3 + O\left(G^4\right)\right]. \tag{6}$$

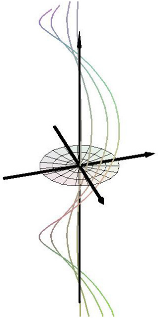
At the zeroth order we recovered the classical relation among mass, charge, external electic field intensity and acceleration.

The electromagnetic magic field



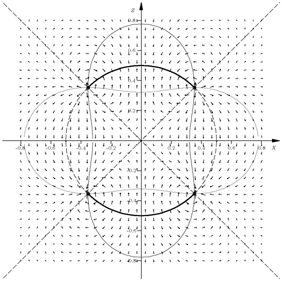
Magic field

Special-relativistic limit of Kerr-Newmann solution is so called electromagnetic magic field – an electromagnetic field of rigidly rotating charged disc surrounded by a rim of opposite charge.
Can serve as a classical model of electron due to its gyromagnetic ratio.


“particle” worldlines

Lynden-Bell, D.: Electromagnetic magic: The relativistically rotating disk. Phys. Rev. D **70**, 105017 (2004)

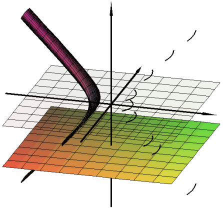
The accelerated electromagnetic magic field



The accelerated electromagnetic magic field

The rotating charged C-metric represents two Kerr-Newmann black holes accelerated uniformly in opposite directions.
Therefore, the special-relativistic limit of rotating charged C-metric leads to two charged rotating discs which are, moreover, bent backwards the direction of acceleration.
Model of accelerated electron?
The 4-current is formally the same as for the electromagnetic magic field

$$\boldsymbol{j} = \frac{1}{2\pi}\frac{q}{a^2x^3}\left(A\boldsymbol{\partial}_t - \frac{1}{a}\,\boldsymbol{\partial}_\phi\right) .$$



Two bent rotating accelerated discs

Bičák, J., Kofroň, D.: Accelerating electromagnetic magic field from the C-metric. Gen. Relativ. Gravit. **41**, 1981-2001 (2009)

The Ehlers frame theory

The Ehlers frame theory is the most rigorous approach to the Newtonian limit of relativistic spacetimes. Fundamental objects of the Ehlers frame theory are

- M – 4 dimensional manifold endowed with affine connection
- t_{ij}, s^{ij}, T^{ij} –time and space metric, stress-energy tensor
- Γ^i_{jk} – torsion-less affine connection with which both space and time metric are compatible
- λ, G – causuality and gravitation constant

$$t_{ij}\, s^{jk} = -\lambda \delta_i^k$$

for non-null λ the causuality constant is identified with $1/c^2$. In the limit $\lambda \to 0$ we recover the Newton-Cartan theory of gravitation.
There is a lot of neccessary and sufficient conditions for the existence of the Newtonian limit of a relativistic spacetime.

From the physical point of view: there has to exist (at least locally) coordinate system in which the metric take the form $t_{ij} = \mathrm{diag}(1,0,0,0)$ and $s^{ij} = \mathrm{diag}(0,1,1,1)$. The Newtoninan gravitational field is then expressed throught the affine connection

$$\Gamma^a_{bc} = t_{,b} t_{,c} s^{ad} \phi_{,d}$$

where ϕ is the Newtonian gravitational potential.

Ehlers, J.: Newtonian Limit of General Relativity. In: J.P. Francoise, G.L. Naber, S.T. Tsou (eds.) Encyclopedia of mathematical physics, vol. 3, pp. 503–509. Elsevier (2006)

The uncharged non-rotating C-metric in the global boost rotation symmetric coordinates

$$\mathrm{d}s^2 = \frac{1}{z^2 - t^2}\left[-e^{\mu}\,(z\mathrm{d}t - t\mathrm{d}z)^2 + e^{\nu}\,(z\mathrm{d}z - t\mathrm{d}t)^2\right] + e^{\nu}\mathrm{d}r^2 + e^{-\mu}r^2\mathrm{d}\phi^2$$

$$\begin{aligned}
R_{\pm} &= \frac{1}{2}\sqrt{\left(r^2 + z^2 - t^2 - \frac{1}{A_{\pm}^2}\right)^2 + \frac{4r^2}{A_{\pm}^2}} \\
\mu &= \frac{1}{2}\ln\frac{R_{+} + R_{-} - \left(\frac{1}{2A_{-}^2} - \frac{1}{2A_{+}^2}\right)}{R_{+} + R_{-} + \left(\frac{1}{2A_{-}^2} - \frac{1}{2A_{+}^2}\right)} \\
\mu + \nu &= \frac{1}{2}\ln\frac{(R_{+} + R_{-})^2 - \left(\frac{1}{2A_{-}^2} - \frac{1}{2A_{+}^2}\right)^2}{4R_{+}R_{-}} + \\
&\quad \frac{1}{2}\ln\frac{(R_{-} + Z_{-})\left[\frac{1}{2}\left(r^2 + z^2 - t^2\right)R_{+} + r^2\left(z^2 - t^2\right) + \frac{1}{2}\left(z^2 - t^2 - r^2\right)Z_{+}\right]}{(R_{+} + Z_{+})\left[\frac{1}{2}\left(r^2 + z^2 - t^2\right)R_{-} + r^2\left(z^2 - t^2\right) + \frac{1}{2}\left(z^2 - t^2 - r^2\right)Z_{-}\right]}
\end{aligned}$$

The Newtonian limit - strategy

The strategy and the technique of calculating the Newtonian limit are, in fact, given by the Ehlers frame theory.
We have to

1. introduce all constants into the metric
2. calculate affine connection
3. do a trick (in the case of the boost-rotation symmetric spacetimes)

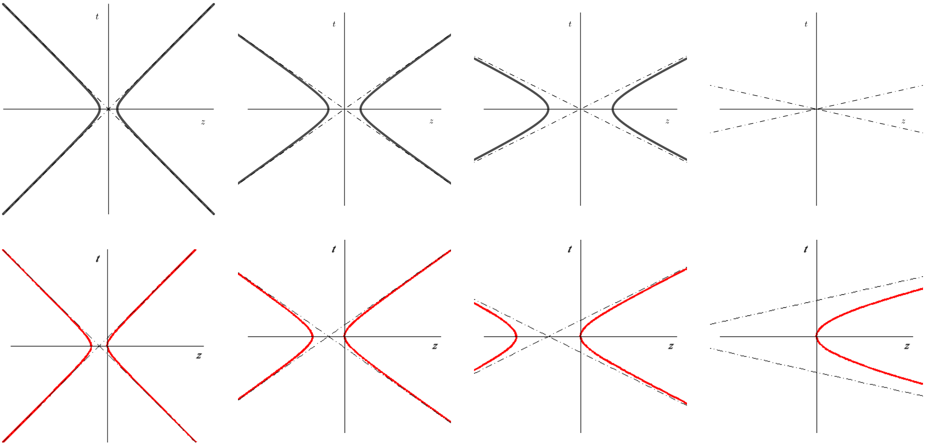
$$z \to z - \frac{1}{\lambda g}\,.$$

4. perform the limit $\lambda \to 0$.

The motivation comes from the hyperbolic motion in special relativity

$$z = \sqrt{\frac{1}{\lambda^2 g^2} + \frac{t^2}{\lambda}}\,.$$

The neccessary trick



$\lambda = 1$ $\lambda = 0.5$ $\lambda = 0.25$ $\lambda = 0.05$

The roof and worldlines of particles for different values of causuality constant λ

Results of the limiting process

After all calculations are done we get the expected (and desired) result
The Newtonian limit of the boost-rotation symmetric vacuum non-rotating spacetimes (with spatially bounded sources) are uniformly accelerated point particles in Newtonian theory with potential

$$\phi = \frac{m}{\sqrt{r^2 + \left(z - \frac{1}{2}gt^2\right)^2}} .$$

Bičák, J., Kofroň, D.: The Newtonian limit of spacetimes for accelerated particles and black holes. Gen. Relativ. Gravit. **41**, 153–172 (2009)

The Newtonian limit of charged non-rotating C-metric

In the previous case we used the advantage of global coordinate system but this is not neccessary.
We can use the purely geometric form of the Ehlers frame theory and perform the Newtonian limit directly with respect to a congruence of observers.
For the charged C-metric which reads

$$\mathrm{d}s^2 = \frac{1}{(x-y)^2}\left[\frac{\mathcal{G}(y)}{\lambda}\,\mathrm{d}t^2 - \frac{1}{A^2\lambda^2\,\mathcal{G}(y)}\,\mathrm{d}y^2 + \frac{1}{A^2\lambda^2\,\mathcal{G}(x)}\,\mathrm{d}x^2 + \frac{\mathcal{G}(x)}{A^2\lambda^2}\,\mathrm{d}\phi^2\right],$$

we find, that the most natural congruence of inertial observers is

$$\boldsymbol{u} = \frac{x-y}{\sqrt{-\mathcal{G}(y)}}\cosh\left(At\sqrt{\lambda}\right)\,\boldsymbol{\partial_t} + A\sqrt{\lambda}\sqrt{-\mathcal{G}(y)}\,\sinh\left(At\sqrt{\lambda}\right)\left[\mathcal{G}(x)\,\boldsymbol{\partial_x} + \mathrm{sgn}\,(1-xy)\,\sqrt{(x-y)^2+\mathcal{G}(x)\mathcal{G}(y)}\,\boldsymbol{\partial_y}\right].$$

The Newtonian limit of charged non-rotating C-metric

Then the gravitational field is given as before and, moreover, we are left with electric and magnetic field

$$\boldsymbol{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{R^3}\left[\rho\,\mathbf{d}\rho + \left(Z - \frac{1}{2}AT^2\right)\mathbf{d}Z\right], \qquad \boldsymbol{B} = -\frac{\mu_0}{4\pi}\frac{qAT}{R^3}\,\boldsymbol{\partial}_\phi\,.$$

These fields are generated by the potentials ($\boldsymbol{E} = \nabla\phi$ and $\boldsymbol{B} = \nabla \times \mathcal{A} = \epsilon^{\alpha\beta\gamma}\nabla_\beta A_\gamma$)

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{q}{R}\,, \qquad \mathcal{A} = -\frac{\mu}{4\pi}\frac{qAT}{R}\,\mathbf{d}z$$

and are solutions of the Newtonian limit of Maxwell equations as proposed by H.P. Künzle:

$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\epsilon_0}\,, \qquad \nabla\times\boldsymbol{E} = 0\,,$$

$$\nabla\cdot\boldsymbol{B} = 0\,, \qquad \nabla\times\boldsymbol{B} = \mu_0\boldsymbol{j} + \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\,.$$

Again, these results shows that we have got uniformly accelerated charged massive point particle as the Newtonian limit of the charged C-metric.

Conclusions

For boost-rotation symmetric spacetimes we have

1. summarized the main properties
2. calculated “equilibrium” conditions
3. provided an interesting special-relativistic limit
4. provided a plausible Newtonian limit (even for charged ones)

Future prospects

1. investigate the “mass problem” using quasilocal definitions


Thank you for attention