On calculating self-force for radiating systems

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AEI Golm

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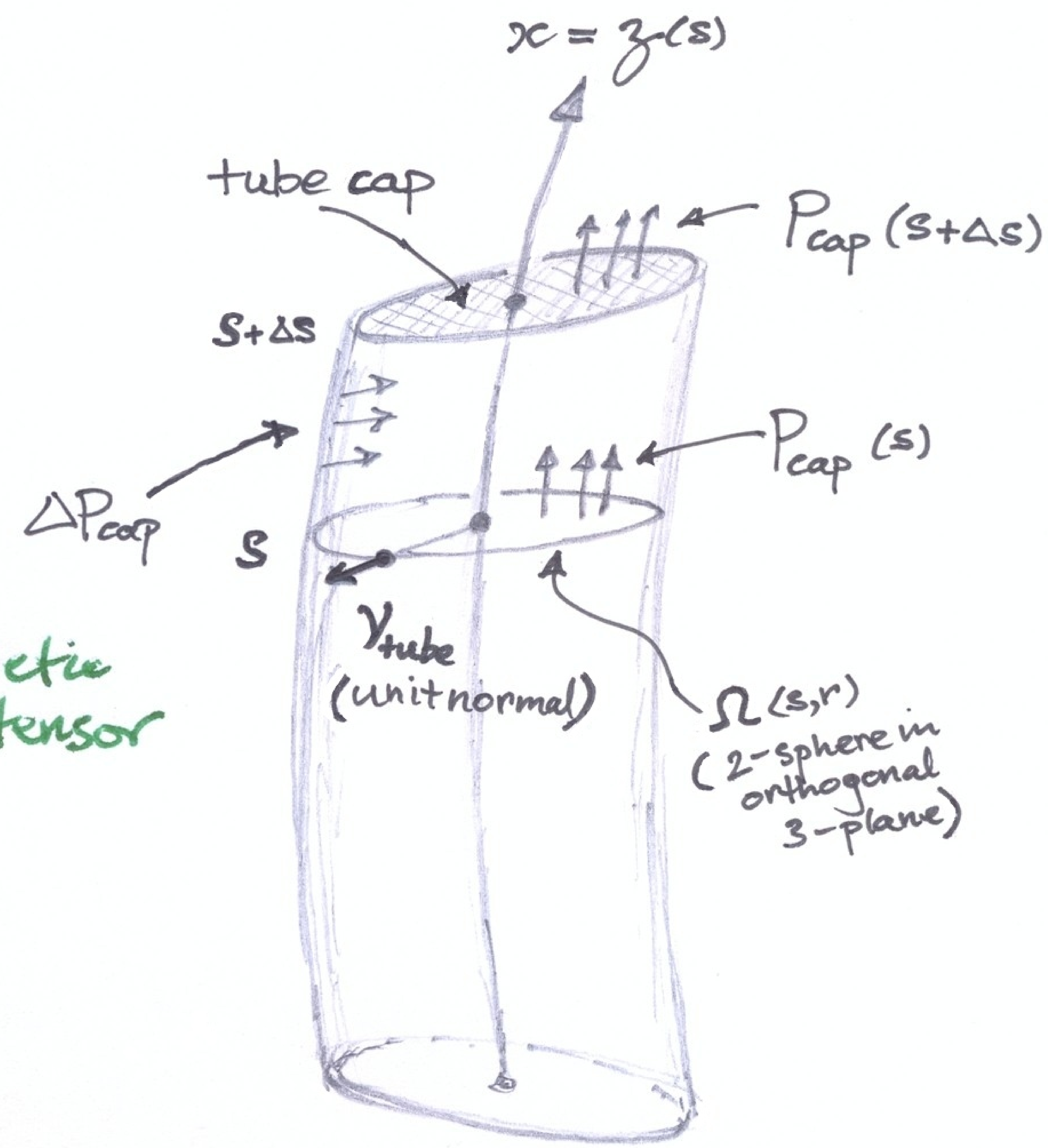
Classical Theory of Radiating Electrons

P. A. M. Dirac

Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences
Vol. 167, No. 929. (Aug. 5, 1938), pp. 148-169.

The idea: Determine the force on the particle by considering energy-momentum balance for a small worldtube centred on the particle worldline:



Electromagnetic stress-energy tensor

$$\frac{dP_{cap}}{ds} = -\frac{1}{c}\int_{\Omega(s,r)} T\cdot\nu_{tube}\, d\Omega$$

Thereby "derive" the

Lorentz-Dirac eqn:
$$\boxed{mc\ddot{z}^{\alpha} = qF^{\alpha}{}_{\beta}\dot{z}^{\beta} + \frac{q^2}{4\pi\epsilon_0 c}\cdot\frac{2}{3}\left(\delta^{\alpha}_{\beta} + \dot{z}^{\alpha}\dot{z}_{\beta}\right)\dddot{z}^{\beta}}$$

Evaluating $\int_\Omega T\cdot\nu\, d\Omega$ in series, in terms of the small tube radius r one finds,

$$\frac{dP_{cap}}{ds} = \frac{a_{-1}(s)}{r} + a_0(s) + a_1(s)\,r + a_2(s)\,r^2 + \cdots$$

← divergent Coulomb field energy

The <u>mass renormalization procedure</u> consists of supposing that $P_{cap}(s,r) = c\,m_B(r)\,u(s)$, and

Bare mass

$$c\,m_B(r) = \frac{-q^2}{8\pi\epsilon_0 c}\,\frac{1}{r} + c\,m,$$

renormalized observable mass

so that the divergent $\frac{1}{r}$ term is cancelled out and replaced by the desired "observable" mass m,

the RHS of the L-D eqn.

$$\frac{d}{ds}(c\,m\,u) = a_0 + a_1 r + \cdots$$

Now take $r \to 0$.

ϵ = tube radius r

What Dirac has to say about this ...

> the energy within the tube must be negative and must tend to $-\infty$ as ϵ tends to zero. This negative energy is needed to compensate for the large positive energy of the Coulomb field just outside the tube, to keep the total energy down to the value appropriate to the rest-mass m. If we want a model of the electron, we must suppose that there is an infinite negative mass at its centre such that, when subtracted from the infinite positive mass of the surrounding Coulomb field, the difference is well defined and is just equal to m.

To carry out the renormalization procedure, the singular term had to be a perfect differential (a derivative w.r.t. s), so that it could be cancelled with a term in $\frac{dP_{cap}}{ds}$, by suitable choice of $P_{cap}(s,r)$.

(1944, 1946) HJ Bhabba and Harish-Chandra:

For any point-multipole particle, the singular terms that arise are always perfect differentials.

... therefore similar renormalization procedures exist for point-multipoles. The L-D eqn corresponds to the simplest case of an electric monopole.

ANNALS OF PHYSICS: 9, 220-259 (1960)

Radiation Damping in a Gravitational Field*

BRYCE S. DEWITT AND ROBERT W. BREHME

we get, in fact,

$$\frac{1}{c}\int_{4\pi} \bar{g}_\mu{}^\alpha\, T^{\mu\nu}\, d\Sigma_\nu$$
$$= \left[\frac{e^2}{2\epsilon\, c^2}\ddot{z}^\alpha - \frac{e^2}{2c}\dot{z}^\beta \int_{-\infty}^{\infty} f^\alpha{}_{\beta\gamma'}\, \dot{z}^{\gamma'}(\tau')\, d\tau' - \frac{e}{c}\bar{F}^{\text{free}\,\alpha}{}_\beta\, \dot{z}^\beta\right] d\tau + O(\epsilon). \tag{5.23}$$

The divergent term in (5.23) has the same kinematical structure as the mass term in Eq. (5.4). It therefore has the effect of an unobservable mass renormalization, and with the introduction of the "observed" mass

$$m = m_0 + \lim_{\epsilon\to 0} \tfrac{1}{2} e^2 \epsilon^{-1} c^{-2}, \tag{5.24}$$

Eq. (5.4) takes the form[11]

$$m\, \ddot{z}^\alpha = ec^{-1}\bar{F}^{\text{free}\,\alpha}{}_\beta\, \dot{z}^\beta + \tfrac{1}{2} e^2 c^{-1} \dot{z}^\beta \int_{-\infty}^{\infty} f^\alpha{}_{\beta\gamma'}\, \dot{z}^{\gamma'}(\tau')\, d\tau'. \tag{5.25}$$

For purposes of application to physically set boundary conditions in the remote past it is more appropriate to work with the field $F^{\text{in}}{}_{\alpha\beta}$. Referring to Eqs. (3.41) and (5.14), we see that Eq. (5.25) then becomes

$$m\, \ddot{z}^\alpha = ec^{-1} F^{\text{in}\,\alpha}{}_\beta\, \dot{z}^\beta + \tfrac{2}{3} e^2 c^{-3}(\dddot{z}^\alpha - c^{-2}\dot{z}^\alpha \ddot{z}^2)$$
$$+ e^2 c^{-1} \dot{z}^\beta \int_{-\infty}^{\tau} f^\alpha{}_{\beta\gamma'}\, \dot{z}^{\gamma'}(\tau')\, d\tau'. \tag{5.26}$$

Footnote...

[11] Equation (5.25) may be regarded as the *definition* of the result of substituting (5.16) in Eq. (3.17) and taking the limit $\epsilon \to 0$.

ANNALS OF PHYSICS: 47, 141–165 (1968)

A Vierbein Formalism of Radiation Damping

J. M. HOBBS

in fact,

$$\int_{4\pi} T^{(\alpha\beta)}\, d\Sigma_{(\beta)} = \Big\{ \frac{e^2}{2\epsilon}\, \ddot{z}^{(\alpha)} - \frac{1}{2}\, e^2 \dot{z}^{(\beta)} \int_{-\infty}^{+\infty} f^{(\alpha)}{}_{(\beta\gamma)} \dot{z}^{(\gamma)}(\tau')\, d\tau'$$
$$- e\bar{F}^{\text{free}(\alpha)}{}_{(\beta)} \dot{z}^{(\beta)} \Big\}\, d\tau + O(\epsilon). \tag{5.24}$$

The divergent term in (5.24) has the same kinematical structure as the mass term in Eq. (5.7). It therefore has the effect of an unobservable mass renormalisation, and with the introduction of the "observed mass"

$$m = m_0 + \lim_{\epsilon\to 0}[\tfrac{1}{2}\epsilon^{-1} e^2], \tag{5.25}$$

Eq. (5.7) takes the form

$$m\ddot{z}^{(\alpha)} = e\bar{F}^{\text{free}(\alpha)}{}_{(\beta)} \dot{z}^{(\beta)} + \frac{1}{2}\, e^2 \dot{z}^{(\beta)} \int_{-\infty}^{+\infty} f^{(\alpha)}{}_{(\beta\gamma)} \dot{z}^{(\gamma)}(\tau')\, d\tau'. \tag{5.26}$$

For the purposes of application to physically set boundary conditions in the remote past it is more appropriate to work with the field $F^{\text{in}}_{(\alpha\beta)}$. Referring to Eqs. (3.15) and (5.17), we see that Eq. (5.26) becomes

$$m\ddot{z}^{(\alpha)} = eF^{\text{in}\,(\alpha)}{}_{(\beta)} \dot{z}^{(\beta)} + \frac{2e^2}{3}\,(\dddot{z}^{(\alpha)} - \dot{z}^{(\alpha)} \ddot{z}^2)$$
$$- \tfrac{1}{3} e^2 R_{\mu\nu} \dot{z}^{\nu} (\mu^{\mu(\alpha)} + \dot{z}^{(\alpha)} \dot{z}^{\mu})$$
$$+ e^2 \dot{z}^{(\beta)} \int_{-\infty}^{\tau} f^{(\alpha)}{}_{(\beta\gamma)} \dot{z}^{(\gamma)}(\tau')\, d\tau'. \tag{5.27}$$

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Gravitational radiation reaction to a particle motion

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we assume that the cap integration for $\epsilon \leqslant \epsilon_0$, which in an unrigorous manner defines the particle momentum $p^{\alpha}_{\epsilon_0}(\tau)$, is proportional to $\dot{z}^{\alpha}(\tau)$:

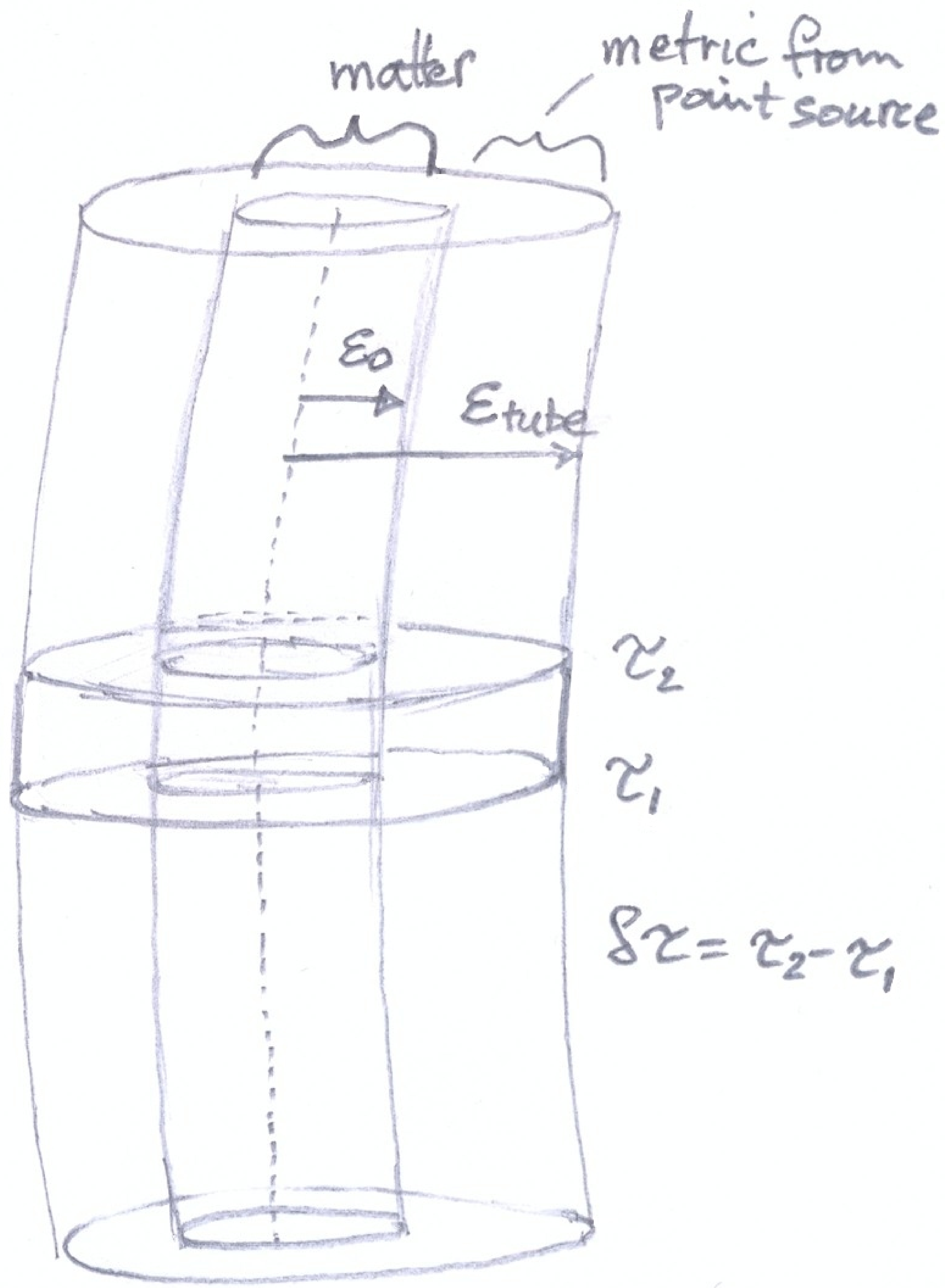
$$\int_{\Sigma_{\mathrm{cap}}(\tau),\epsilon<\epsilon_0} d\Sigma_{\nu}(x) \bar{g}^{\bar{\alpha}}{}_{\beta}(z(\bar{\tau}),z(\tau_x)) \bar{g}^{\beta}{}_{\mu}(z(\tau_x),x) T^{\mu\nu}[\mathbf{h}](x)$$

$$= m(\epsilon_0,\tau) \bar{g}^{\bar{\alpha}}{}_{\beta}(z(\bar{\tau}),z(\tau)) \dot{z}^{\beta}(\tau) + Gm^2 O(m,\tau_r^{-1},\epsilon_{\mathrm{tube}}).$$

Outside the radius ϵ_0, the metric is approximated by that induced by a pointlike particle, Eq. (2.20). Then the cap integration for $\epsilon_0 \leqslant \epsilon \leqslant \epsilon_{\mathrm{tube}}$ is evaluated by using Eq. (3.18) to give

$$\int_{\Sigma_{\mathrm{cap}}(\tau),\epsilon>\epsilon_0} d\Sigma_{\nu}(x) \bar{g}^{\bar{\alpha}}{}_{\beta}(z(\bar{\tau}),z(\tau_x)) \bar{g}^{\beta}{}_{\mu}(z(\tau_x),x) T^{\mu\nu}[\mathbf{h}](x)$$

$$= -\frac{7}{2} Gm^2 \left[\left(\frac{1}{\epsilon_0} - \frac{1}{\epsilon_{\mathrm{tube}}} \right) \bar{g}^{\bar{\alpha}}{}_{\beta}(z(\bar{\tau}),z(\tau)) \dot{z}^{\beta}(\tau) + O(\tau_r^{-1},\epsilon_{\mathrm{tube}}) \right]. \quad (3.23)$$



Thus setting

$$m(\epsilon_0,\tau)=m(\tau)+\frac{7}{2}\frac{Gm^2}{\epsilon_0}, \tag{3.24}$$

Mass renormalization for the inner tube of radius ϵ_0.

we obtain

$$\int_{\Sigma_{\mathrm{cap}}(\tau_2)-\Sigma_{\mathrm{cap}}(\tau_1)} d\Sigma_\nu(x)\, g^{\bar\alpha}{}_\beta(z(\bar\tau),z(\tau_x))$$
$$\times g^\beta{}_\mu(z(\tau_x),x)T^{\mu\nu}[\mathbf{h}](x)$$
$$=\left[\left\{m(\bar\tau)+\frac{7}{2}\frac{Gm^2}{\epsilon_{\mathrm{tube}}}\right\}\ddot z^{\bar\alpha}(\bar\tau)+\dot m(\bar\tau)\dot z^{\bar\alpha}(\bar\tau)\right]\delta\tau+O(\delta\tau^2). \tag{3.25}$$

The integrals over the caps.

these cancel

$$\int_{\Sigma_{\mathrm{tube}}} d\Sigma_\nu(x)\, g^{\bar\alpha}{}_\beta(z(\bar\tau),z(\tau_x))\,g^\beta{}_\mu(z(\tau_x),x)T^{\mu\nu}[\mathbf{h}](x)$$
$$=Gm^2\left\{\left(\left(-\frac{7}{2\epsilon_{\mathrm{tube}}}\ddot z^{\bar\alpha}-\frac{2}{3}\dot z^{\bar\alpha}V_{\bar\beta\bar\gamma\bar\delta}\dot z^{\bar\beta}\dot z^{\bar\gamma}\dot z^{\bar\delta}\right.\right.\right.$$
$$-\frac{2}{3}\dot z^{\bar\alpha}V_{\bar\beta}\dot z^{\bar\beta}+(V^{\bar\alpha}{}_{\bar\beta\bar\gamma}+V^{\bar\alpha}{}_{\bar\gamma\bar\beta}-V_{\bar\beta\bar\gamma}{}^{\bar\alpha})\dot z^{\bar\beta}\dot z^{\bar\gamma}$$
$$\left.\left.-\frac{1}{2}V^{\bar\alpha}\right)(\bar\tau)+O(\tau_r^{-1},\epsilon_{\mathrm{tube}})\right\}\delta\tau+O(\delta\tau^2). \tag{3.21}$$

The integral over length δz of the tube

The MiSaTaQuWa formula

Quinn + Wald '97.

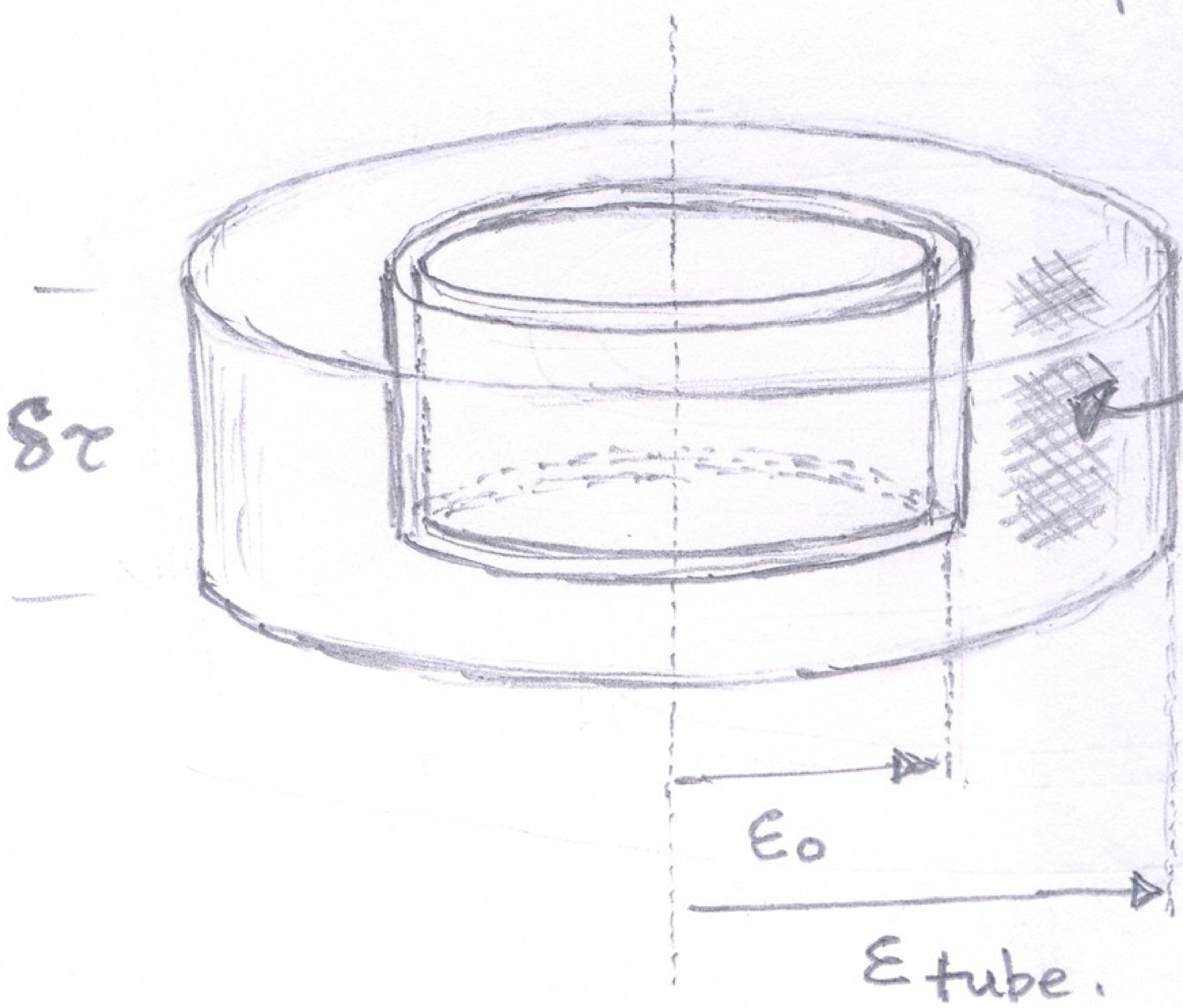
Eqn of motion: $0=$ [cap integral (3·25)] + [tube integral (3·21)]

- The picture is starting to look nicer — no field singularity.

The field of the point source *is* singular, but is only used at radius $\geq \epsilon_0$.

- However, introduction of the outer tube at radius ϵ_{tube} has achieved nothing — the renormalization procedure is exactly the same as that used in the electromagnetic self-force calculations. It could have been done with just the inner tube:

Split the volume integration into 2 regions.



For the anular region, $\nabla \cdot T \equiv 0$ by construction, because the field here is that of a pt source solution.

[This is why the $\frac{1}{\epsilon_{tube}}$ terms cancel.]

- An ∞ mass renormalization has not been avoided.

ϵ_0 can not be left small but not zero, and it is not legitimate to simply drop terms $O(\epsilon_0)$. If it were legitimate to do so, then keeping these terms would give an equation of motion that is more accurate (not less). But in fact, we would get an equation of motion that depended on the arbitrary radius of an imaginary tube that was simply used to facilitate some calculation.

A. Norton
CQG '09

There is a tube method that produces energy-momentum balance equations with ϵ_0 eliminated at all orders, including the $\frac{1}{\epsilon_0}$ term.

... and classical mass renormalization does "work".

How does one derive an equation of motion for an extended particle-like object in a classical field theory?

Extended Objects:

black holes | topological solitons eg. Skyrmions | classical electron models eg. charged membrane | Multipole sources in Maxwell electrodynamics

Non-linear field theory | Linear field + structure (effectively nonlinear) | Not specified

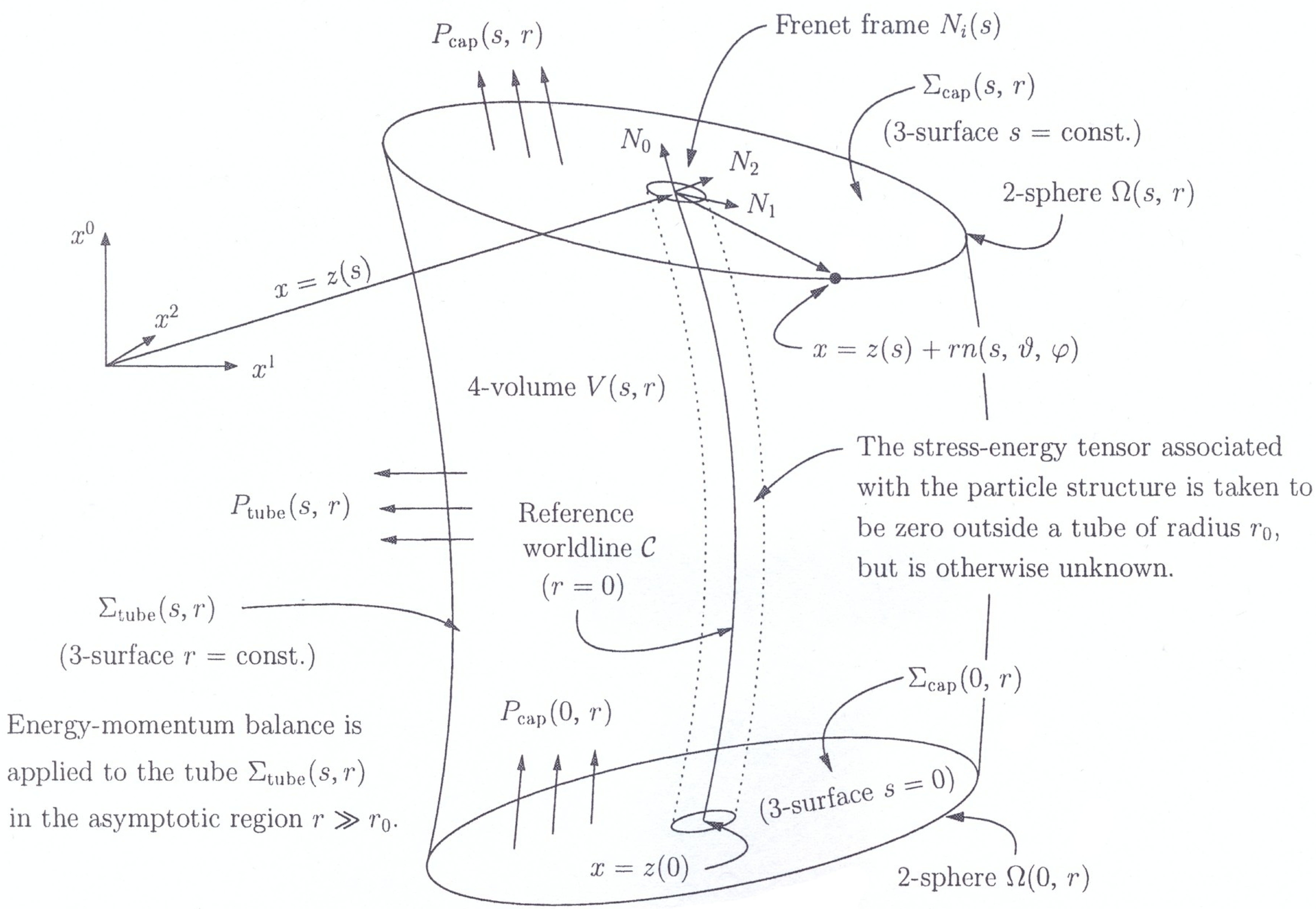
Suppose $\exists$ a conserved stress-energy tensor T,

$$\nabla \cdot T = 0,$$

and that the particle has finite extent, so that for some $r_0 > 0$ we have

$$T = \begin{cases} T_{\text{matter}} & r < r_0 \\ T_{\text{Linear}} & r > r_0 \end{cases}$$

where T_{Linear} is the stress-energy tensor for the linearized theory.



Integrate $\nabla \cdot T = 0$ over the tube volume $V(s,r)$ and use the divergence theorem to see that

$$0 = \int_{V(s,r)} \nabla \cdot T \, dV \quad \Longrightarrow \quad P_{\rm cap}(s,r) = P_{\rm cap}(0,r) - P_{\rm tube}(s,r) .$$

$$P_{cap}(s,r) = P_{cap}(0,r) - P_{tube}(s,r)$$

$$= P_{cap}(0,r) - \frac{1}{c}\int_{s'=0}^{s}\int_{\Omega(s',r)} T\cdot\nu_{tube}\, d\Omega\, ds'$$

Also,

$$P_{cap}(s,r) = \frac{1}{c}\int_{r'=0}^{r}\int_{\Omega(s,r')} T\cdot\nu_{cap}\, d\Omega\, dr'$$

$\Omega(s,r)$ is the 2-sphere of radius r centred at $z(s)$ in the 3-plane orthogonal to $\dot{z}(s)$. $d\Omega = r^2 \sin\theta\, d\theta\, d\phi$.

In general T is not known for $r<r_0$. Therefore neither of the above expressions for $P_{cap}(s,r)$ can be evaluated. Nevertheless, for $r>r_0$ the partial derivatives <u>can</u> be evaluated:

$$\frac{\partial P_{cap}}{\partial s} = -\frac{1}{c}\int_{\Omega(s,r)} T\cdot\nu_{tube}\, d\Omega$$

$$\frac{\partial P_{cap}}{\partial r} = \frac{1}{c}\int_{\Omega(s,r)} T\cdot\nu_{cap}\, d\Omega$$

These integrals are over a sphere of radius r. For $r>r_0$ we have $T = T_{Linear}$.

After replacing T by T_{Linear} the integrals can be evaluated as series in r,

$$\frac{\partial P_{cap}}{\partial s} = (\text{power series in } r)$$

For a monopole this series starts at $\frac{1}{r}$. For a dipole, at $\frac{1}{r^3}$.

$$* \quad \frac{\partial P_{cap}}{\partial r} = (\text{power series in } r)$$

Assuming convergence, integrate $*$ to get

The integration constant will be identified as the momentum of the particle.

$$P_{cap}(s,r) = p(s) + (\text{power series in } r).$$

Now differentiate w.r.t. s to get a 2nd expression for $\frac{\partial P_{cap}}{\partial s}$,

$$\frac{\partial P_{cap}}{\partial s} = \dot{p}(s) + (\text{power series in } r).$$

Subtract the two expressions to obtain

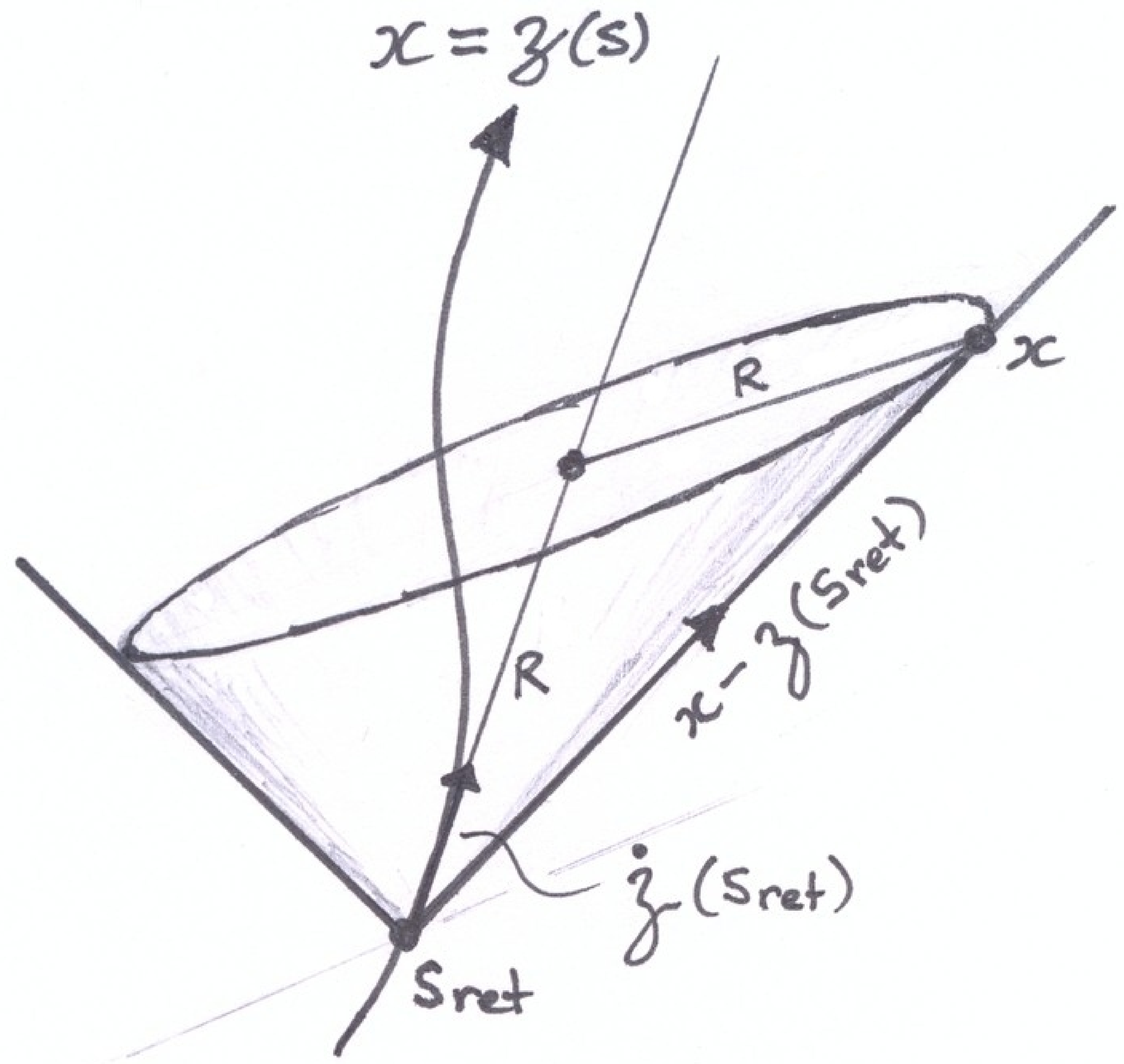
$$0 = \dot{p}(s) + (\text{power series in } r).$$

Since r is arbitrary, all of the coefficients in this series must vanish. The coefficients of r^n, $n \neq 0$, vanish identically. For $n=0$ we get an equation of the form

$$\dot{p}(s) = f. \qquad \leftarrow \text{the balance equation.}$$

Example Equation of motion of a charged particle in Minkowski spacetime

Assume the far-field ($r \gg r_0$) of the particle is well approximated by that of a point-multipole source. We consider the simplest case, an electric monopole. The far-field of the particle is then given by the retarded Liénard-Wiechert potential for a point charge q,



$$A(x) = \frac{q}{4\pi\epsilon_0 c}\left[\frac{\dot{z}}{R}\right]_{s=s_{ret}}$$

where

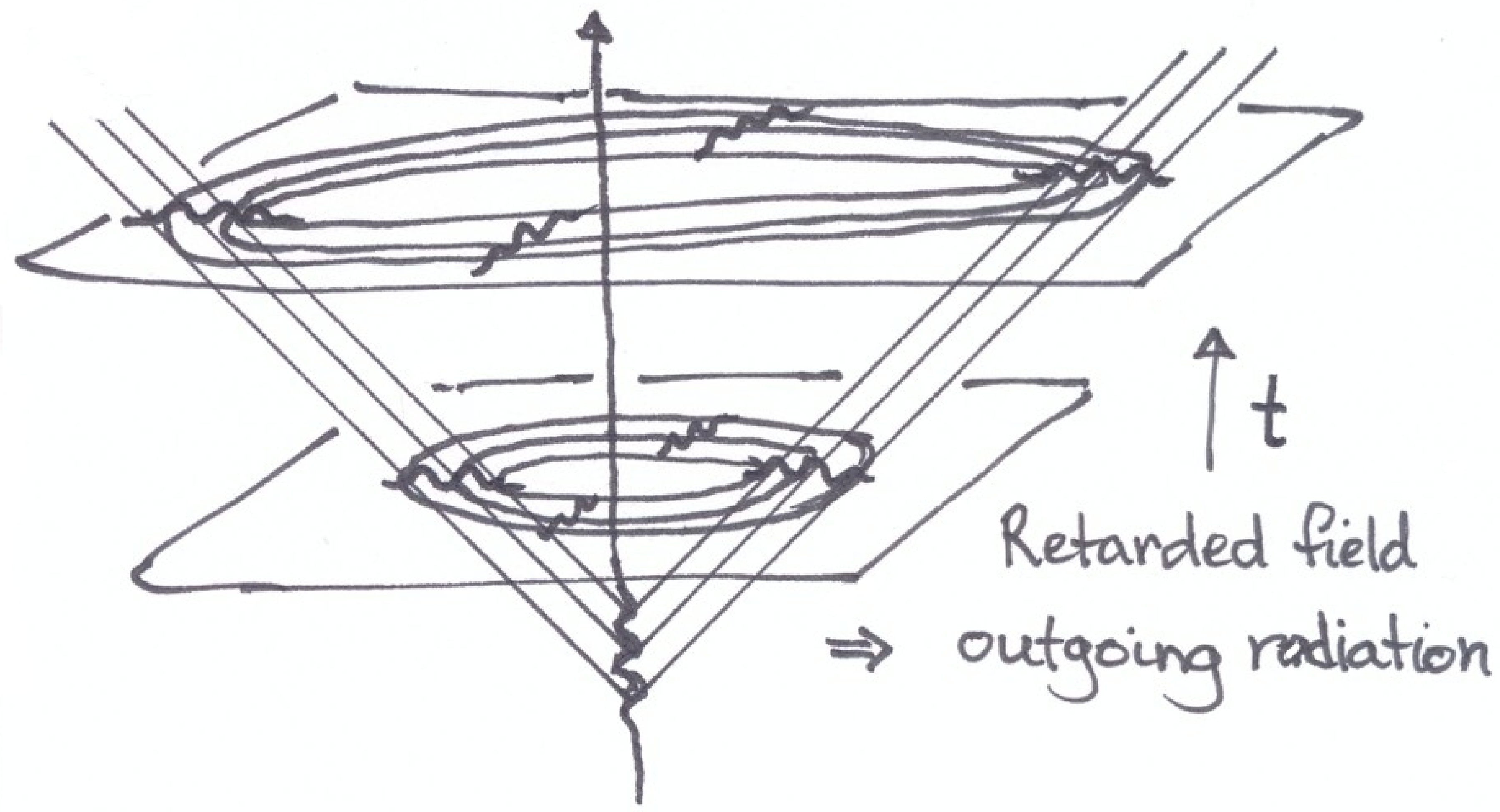
$$R_{ret} = -\dot{z}_\alpha(s_{ret})\left(x^\alpha - z^\alpha(s_{ret})\right).$$

$A(x)$ is a solution, for arbitrary $z(s)$, of the Maxwell equations,

$$F^{\alpha\beta}{}_{|\beta} = \mu_0 J^\alpha$$

for a point source with current

$$J^\alpha(x) = cq\int_{-\infty}^{\infty} \dot{z}^\alpha(s)\,\delta^4(x - z(s))\,ds.$$



The electromagnetic field corresponding to the Lienard-Wiechert potential is

$$F_{L\text{-}W\,\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}$$

$$= \frac{q}{4\pi\epsilon_0 c}\left[(\dot{z}_\alpha k_\beta - \dot{z}_\beta k_\alpha)(1+aR)R^{-2} + (\ddot{z}_\alpha k_\beta - \ddot{z}_\beta k_\alpha)R^{-1}\right]_{ret}$$

where k is the null vector $k = (x - z(s_{ret}))R^{-1}$, so that $\dot{z}_{ret} \cdot k = -1$, and a is the acceleration dependent scalar $a = \ddot{z}_{ret} \cdot k$.

[Partial derivatives of retarded fields can be calculated as

$$(\cdot)_{,\beta} = \frac{\partial}{\partial x^\beta} - k_\beta \frac{\partial}{\partial s_{ret}},$$

which follows from differentiating the null constraint $(x - z(s_{ret}))^2 = 0$.]

The electromagnetic field used to define the stress-energy tensor is taken to be

$$F_{tot} = F_{L\text{-}W} + F$$

where F is any background field that satisfies the source-free Maxwell equations throughout the spacetime region occupied by the particle. Then,

$$T_{Linear}^{\alpha\beta} = \frac{1}{\mu_0}\left(F_{tot}^{\alpha\mu} F_{tot}^{\beta\nu}\eta_{\mu\nu} - \frac{1}{4}\eta^{\alpha\beta} F_{tot}^{\mu\nu} F_{tot\,\mu\nu}\right)$$

The following series are obtained in terms of the Frenet frame of the worldline. The frame satisfies the Frenet-Serret equations,

$$\begin{aligned}
\dot{N}_0 &= \kappa_1 N_1 \\
\dot{N}_1 &= \kappa_1 N_0 + \kappa_2 N_2 \\
\dot{N}_2 &= -\kappa_2 N_1 + \kappa_3 N_3 \\
\dot{N}_3 &= -\kappa_3 N_2 .
\end{aligned}$$

Here $N_0 = \dot{z}$ is the unit tangent, and $\kappa_1, \kappa_2, \kappa_3$, are the Frenet curvatures.

Series expression for dPcap/ds

In[214]:= **dPds[s, r]**

Out[214]=

$$-\frac{q^2 N_1[s]\,\kappa_1[s]}{8\,(c\,\pi\,\epsilon_0)\,r}+\Bigg(\frac{q^2 N_2[s]\,\kappa_1[s]\,\kappa_2[s]}{6\,c\,\pi\,\epsilon_0}+\frac{q^2 N_1[s]\,(\kappa_1)'[s]}{6\,c\,\pi\,\epsilon_0}+q\,N_1[s]\,F^{01}[s]+q\,N_2[s]\,F^{02}[s]+q\,N_3[s]\,F^{03}[s]\Bigg)+$$

$$\Bigg(\frac{q^2 N_1[s]\,\kappa_1[s]^3}{16\,c\,\pi\,\epsilon_0}+\frac{q^2 N_1[s]\,\kappa_1[s]\,\kappa_2[s]^2}{12\,c\,\pi\,\epsilon_0}-\frac{q^2 N_3[s]\,\kappa_1[s]\,\kappa_2[s]\,\kappa_3[s]}{12\,c\,\pi\,\epsilon_0}+$$

$$\frac{q^2 N_0[s]\,\kappa_1[s]\,(\kappa_1)'[s]}{24\,c\,\pi\,\epsilon_0}-\frac{q^2 N_2[s]\,\kappa_2[s]\,(\kappa_1)'[s]}{6\,c\,\pi\,\epsilon_0}-\frac{q^2 N_2[s]\,\kappa_1[s]\,(\kappa_2)'[s]}{12\,c\,\pi\,\epsilon_0}-\frac{q^2 N_1[s]\,(\kappa_1)''[s]}{12\,c\,\pi\,\epsilon_0}\Bigg)\,r+$$

$$\Bigg(-\frac{q^2 N_0[s]\,\kappa_1[s]^2\,\kappa_2[s]^2}{36\,c\,\pi\,\epsilon_0}-\frac{q^2 N_2[s]\,\kappa_1[s]\,\kappa_2[s]^3}{36\,c\,\pi\,\epsilon_0}-\frac{q^2 N_2[s]\,\kappa_1[s]\,\kappa_2[s]\,\kappa_3[s]^2}{36\,c\,\pi\,\epsilon_0}-\frac{q^2 N_1[s]\,\kappa_1[s]^2\,(\kappa_1)'[s]}{18\,c\,\pi\,\epsilon_0}-\frac{q^2 N_1[s]\,\kappa_2[s]^2\,(\kappa_1)'[s]}{12\,c\,\pi\,\epsilon_0}+$$

$$\frac{q^2 N_3[s]\,\kappa_2[s]\,\kappa_3[s]\,(\kappa_1)'[s]}{12\,c\,\pi\,\epsilon_0}-\frac{q^2 N_0[s]\,(\kappa_1)'[s]^2}{18\,c\,\pi\,\epsilon_0}-\frac{q^2 N_1[s]\,\kappa_1[s]\,\kappa_2[s]\,(\kappa_2)'[s]}{12\,c\,\pi\,\epsilon_0}+\frac{q^2 N_3[s]\,\kappa_1[s]\,\kappa_3[s]\,(\kappa_2)'[s]}{18\,c\,\pi\,\epsilon_0}+$$

$$\frac{q^2 N_2[s]\,(\kappa_1)'[s]\,(\kappa_2)'[s]}{12\,c\,\pi\,\epsilon_0}+\frac{q^2 N_3[s]\,\kappa_1[s]\,\kappa_2[s]\,(\kappa_3)'[s]}{36\,c\,\pi\,\epsilon_0}-\frac{q^2 N_0[s]\,\kappa_1[s]\,(\kappa_1)''[s]}{36\,c\,\pi\,\epsilon_0}+\frac{q^2 N_2[s]\,\kappa_2[s]\,(\kappa_1)''[s]}{12\,c\,\pi\,\epsilon_0}+$$

$$\frac{q^2 N_2[s]\,\kappa_1[s]\,(\kappa_2)''[s]}{36\,c\,\pi\,\epsilon_0}+\frac{q^2 N_1[s]\,(\kappa_1)^{(3)}[s]}{36\,c\,\pi\,\epsilon_0}-\frac{1}{3}q\,N_1[s]\,\kappa_1[s]^2\,F^{01}[s]-\frac{1}{3}q\,N_0[s]\,(\kappa_1)'[s]\,F^{01}[s]-\frac{1}{3}q\,N_2[s]\,\kappa_1[s]^2\,F^{02}[s]-$$

$$\frac{1}{3}q\,N_0[s]\,\kappa_1[s]\,\kappa_2[s]\,F^{02}[s]-\frac{1}{3}q\,N_3[s]\,\kappa_1[s]^2\,F^{03}[s]-\frac{1}{3}q\,N_1[s]\,\kappa_1[s]\,\kappa_2[s]\,F^{12}[s]+\frac{1}{3}q\,N_2[s]\,(\kappa_1)'[s]\,F^{12}[s]+$$

$$\frac{1}{3}q\,N_3[s]\,\kappa_1[s]\,\kappa_2[s]\,F^{23}[s]-\frac{1}{3}q\,N_3[s]\,(\kappa_1)'[s]\,F^{31}[s]+\frac{1}{6}q\,N_2[s]\,\kappa_1[s]\,{F^{01}}_2[s]+\frac{1}{6}q\,N_3[s]\,\kappa_1[s]\,{F^{01}}_3[s]-$$

$$\frac{1}{6}q\,N_1[s]\,\kappa_1[s]\,{F^{02}}_2[s]-\frac{1}{6}q\,N_1[s]\,\kappa_1[s]\,{F^{03}}_3[s]-\frac{1}{6}q\,N_0[s]\,\kappa_1[s]\,{F^{12}}_2[s]+\frac{1}{6}q\,N_0[s]\,\kappa_1[s]\,{F^{31}}_3[s]\Bigg)\,r^2+$$

$$\Bigg(\frac{q^2 N_1[s]\,\kappa_1[s]^5}{128\,c\,\pi\,\epsilon_0}+\frac{31\,q^2 N_1[s]\,\kappa_1[s]^3\,\kappa_2[s]^2}{540\,c\,\pi\,\epsilon_0}-\frac{q^2 N_1[s]\,\kappa_1[s]\,\kappa_2[s]^4}{144\,c\,\pi\,\epsilon_0}-\frac{71\,q^2 N_3[s]\,\kappa_1[s]^3\,\kappa_2[s]\,\kappa_3[s]}{1440\,c\,\pi\,\epsilon_0}+\frac{q^2 N_3[s]\,\kappa_1[s]\,\kappa_2[s]^3\,\kappa_3[s]}{144\,c\,\pi\,\epsilon_0}-$$

$$\frac{q^2 N_1[s]\,\kappa_1[s]\,\kappa_2[s]^2\,\kappa_3[s]^2}{144\,c\,\pi\,\epsilon_0}+\frac{q^2 N_3[s]\,\kappa_1[s]\,\kappa_2[s]\,\kappa_3[s]^3}{144\,c\,\pi\,\epsilon_0}+\frac{3\,q^2 N_0[s]\,\kappa_1[s]^3\,(\kappa_1)'[s]}{160\,c\,\pi\,\epsilon_0}-\frac{77\,q^2 N_2[s]\,\kappa_1[s]^2\,\kappa_2[s]\,(\kappa_1)'[s]}{480\,c\,\pi\,\epsilon_0}+$$

$$\frac{q^2 N_0[s]\,\kappa_1[s]\,\kappa_2[s]^2\,(\kappa_1)'[s]}{27\,c\,\pi\,\epsilon_0}+\frac{q^2 N_2[s]\,\kappa_2[s]^3\,(\kappa_1)'[s]}{36\,c\,\pi\,\epsilon_0}+\frac{q^2 N_2[s]\,\kappa_2[s]\,\kappa_3[s]^2\,(\kappa_1)'[s]}{36\,c\,\pi\,\epsilon_0}+\frac{q^2 N_1[s]\,\kappa_1[s]\,(\kappa_1)'[s]^2}{2160\,c\,\pi\,\epsilon_0}-$$

$$\frac{71\,q^2 N_2[s]\,\kappa_1[s]^3\,(\kappa_2)'[s]}{1440\,c\,\pi\,\epsilon_0}+\frac{q^2 N_0[s]\,\kappa_1[s]^2\,\kappa_2[s]\,(\kappa_2)'[s]}{27\,c\,\pi\,\epsilon_0}+\frac{q^2 N_2[s]\,\kappa_1[s]\,\kappa_2[s]^2\,(\kappa_2)'[s]}{24\,c\,\pi\,\epsilon_0}+\frac{q^2 N_2[s]\,\kappa_1[s]\,\kappa_3[s]^2\,(\kappa_2)'[s]}{48\,c\,\pi\,\epsilon_0}+$$

$$\frac{q^2 N_1[s]\,\kappa_2[s]\,(\kappa_1)'[s]\,(\kappa_2)'[s]}{12\,c\,\pi\,\epsilon_0} - \frac{q^2 N_3[s]\,\kappa_3[s]\,(\kappa_1)'[s]\,(\kappa_2)'[s]}{18\,c\,\pi\,\epsilon_0} + \frac{q^2 N_1[s]\,\kappa_1[s]\,(\kappa_2)'[s]^2}{48\,c\,\pi\,\epsilon_0} + \frac{q^2 N_2[s]\,\kappa_1[s]\,\kappa_2[s]\,\kappa_3[s]\,(\kappa_3)'[s]}{48\,c\,\pi\,\epsilon_0} -$$

$$\frac{q^2 N_3[s]\,\kappa_2[s]\,(\kappa_1)'[s]\,(\kappa_3)'[s]}{36\,c\,\pi\,\epsilon_0} - \frac{q^2 N_3[s]\,\kappa_1[s]\,(\kappa_2)'[s]\,(\kappa_3)'[s]}{48\,c\,\pi\,\epsilon_0} + \frac{7\,q^2 N_1[s]\,\kappa_1[s]^2\,(\kappa_1)''[s]}{1440\,c\,\pi\,\epsilon_0} + \frac{q^2 N_1[s]\,\kappa_2[s]^2\,(\kappa_1)''[s]}{24\,c\,\pi\,\epsilon_0} -$$

$$\frac{q^2 N_3[s]\,\kappa_2[s]\,\kappa_3[s]\,(\kappa_1)''[s]}{24\,c\,\pi\,\epsilon_0} + \frac{59\,q^2 N_0[s]\,(\kappa_1)'[s]\,(\kappa_1)''[s]}{864\,c\,\pi\,\epsilon_0} - \frac{q^2 N_2[s]\,(\kappa_2)'[s]\,(\kappa_1)''[s]}{24\,c\,\pi\,\epsilon_0} + \frac{q^2 N_1[s]\,\kappa_1[s]\,\kappa_2[s]\,(\kappa_2)''[s]}{36\,c\,\pi\,\epsilon_0} -$$

$$\frac{q^2 N_3[s]\,\kappa_1[s]\,\kappa_3[s]\,(\kappa_2)''[s]}{48\,c\,\pi\,\epsilon_0} - \frac{q^2 N_2[s]\,(\kappa_1)'[s]\,(\kappa_2)''[s]}{36\,c\,\pi\,\epsilon_0} - \frac{q^2 N_3[s]\,\kappa_1[s]\,\kappa_2[s]\,(\kappa_3)''[s]}{144\,c\,\pi\,\epsilon_0} + \frac{q^2 N_0[s]\,\kappa_1[s]\,(\kappa_1)^{(3)}[s]}{96\,c\,\pi\,\epsilon_0} -$$

$$\frac{q^2 N_2[s]\,\kappa_2[s]\,(\kappa_1)^{(3)}[s]}{36\,c\,\pi\,\epsilon_0} - \frac{q^2 N_2[s]\,\kappa_1[s]\,(\kappa_2)^{(3)}[s]}{144\,c\,\pi\,\epsilon_0} - \frac{q^2 N_1[s]\,(\kappa_1)^{(4)}[s]}{144\,c\,\pi\,\epsilon_0} - \frac{2}{9}\,q\,N_0[s]\,\kappa_1[s]\,\kappa_2[s]^2\,F^{01}[s] +$$

$$\frac{2}{9}\,q\,N_1[s]\,\kappa_1[s]\,(\kappa_1)'[s]\,F^{01}[s] + \frac{2}{9}\,q\,N_0[s]\,(\kappa_1)''[s]\,F^{01}[s] + \frac{2}{3}\,c\,\pi\,\epsilon_0\,N_1[s]\,\kappa_1[s]\,F^{01}[s]^2 + \frac{2}{9}\,q\,N_2[s]\,\kappa_1[s]\,(\kappa_1)'[s]\,F^{02}[s] +$$

$$\frac{4}{9}\,q\,N_0[s]\,\kappa_2[s]\,(\kappa_1)'[s]\,F^{02}[s] + \frac{2}{9}\,q\,N_0[s]\,\kappa_1[s]\,(\kappa_2)'[s]\,F^{02}[s] + \frac{4}{3}\,c\,\pi\,\epsilon_0\,N_2[s]\,\kappa_1[s]\,F^{01}[s]\,F^{02}[s] - \frac{2}{3}\,c\,\pi\,\epsilon_0\,N_1[s]\,\kappa_1[s]\,F^{02}[s]^2 +$$

$$\frac{2}{9}\,q\,N_0[s]\,\kappa_1[s]\,\kappa_2[s]\,\kappa_3[s]\,F^{03}[s] + \frac{2}{9}\,q\,N_3[s]\,\kappa_1[s]\,(\kappa_1)'[s]\,F^{03}[s] + \frac{4}{3}\,c\,\pi\,\epsilon_0\,N_3[s]\,\kappa_1[s]\,F^{01}[s]\,F^{03}[s] -$$

$$\frac{2}{3}\,c\,\pi\,\epsilon_0\,N_1[s]\,\kappa_1[s]\,F^{03}[s]^2 + \frac{2}{9}\,q\,N_2[s]\,\kappa_1[s]\,\kappa_2[s]^2\,F^{12}[s] + \frac{4}{9}\,q\,N_1[s]\,\kappa_2[s]\,(\kappa_1)'[s]\,F^{12}[s] + \frac{2}{9}\,q\,N_1[s]\,\kappa_1[s]\,(\kappa_2)'[s]\,F^{12}[s] -$$

$$\frac{2}{9}\,q\,N_2[s]\,(\kappa_1)''[s]\,F^{12}[s] - \frac{4}{3}\,c\,\pi\,\epsilon_0\,N_0[s]\,\kappa_1[s]\,F^{02}[s]\,F^{12}[s] - \frac{2}{3}\,c\,\pi\,\epsilon_0\,N_1[s]\,\kappa_1[s]\,F^{12}[s]^2 + \frac{2}{9}\,q\,N_2[s]\,\kappa_1[s]\,\kappa_2[s]\,\kappa_3[s]\,F^{23}[s] -$$

$$\frac{4}{9}\,q\,N_3[s]\,\kappa_2[s]\,(\kappa_1)'[s]\,F^{23}[s] - \frac{2}{9}\,q\,N_3[s]\,\kappa_1[s]\,(\kappa_2)'[s]\,F^{23}[s] + \frac{4}{3}\,c\,\pi\,\epsilon_0\,N_3[s]\,\kappa_1[s]\,F^{12}[s]\,F^{23}[s] +$$

$$\frac{2}{3}\,c\,\pi\,\epsilon_0\,N_1[s]\,\kappa_1[s]\,F^{23}[s]^2 - \frac{2}{9}\,q\,N_3[s]\,\kappa_1[s]\,\kappa_2[s]^2\,F^{31}[s] - \frac{2}{9}\,q\,N_1[s]\,\kappa_1[s]\,\kappa_2[s]\,\kappa_3[s]\,F^{31}[s] + \frac{2}{9}\,q\,N_3[s]\,(\kappa_1)''[s]\,F^{31}[s] +$$

$$\frac{4}{3} c \pi \epsilon_0 N_0[s]\, \kappa_1[s]\, F^{03}[s]\, F^{31}[s] + \frac{4}{3} c \pi \epsilon_0 N_2[s]\, \kappa_1[s]\, F^{23}[s]\, F^{31}[s] - \frac{2}{3} c \pi \epsilon_0 N_1[s]\, \kappa_1[s]\, F^{31}[s]^2 +$$

$$\frac{2}{9} q N_2[s]\, \kappa_1[s]\, \kappa_2[s]\, F^{01}{}_1[s] + \frac{2}{9} q N_1[s]\, (\kappa_1)'[s]\, F^{01}{}_1[s] + \frac{4}{3} c \pi \epsilon_0 N_1[s]\, F^{01}[s]\, F^{01}{}_1[s] + \frac{4}{3} c \pi \epsilon_0 N_2[s]\, F^{02}[s]\, F^{01}{}_1[s] +$$

$$\frac{4}{3} c \pi \epsilon_0 N_3[s]\, F^{03}[s]\, F^{01}{}_1[s] + \frac{2}{9} q N_1[s]\, \kappa_1[s]\, \kappa_2[s]\, F^{01}{}_2[s] - \frac{2}{9} q N_2[s]\, (\kappa_1)'[s]\, F^{01}{}_2[s] - \frac{4}{3} c \pi \epsilon_0 N_2[s]\, F^{01}[s]\, F^{01}{}_2[s] +$$

$$\frac{4}{3} c \pi \epsilon_0 N_1[s]\, F^{02}[s]\, F^{01}{}_2[s] + \frac{4}{3} c \pi \epsilon_0 N_0[s]\, F^{12}[s]\, F^{01}{}_2[s] - \frac{2}{9} q N_3[s]\, (\kappa_1)'[s]\, F^{01}{}_3[s] - \frac{4}{3} c \pi \epsilon_0 N_3[s]\, F^{01}[s]\, F^{01}{}_3[s] +$$

$$\frac{4}{3} c \pi \epsilon_0 N_1[s]\, F^{03}[s]\, F^{01}{}_3[s] - \frac{4}{3} c \pi \epsilon_0 N_0[s]\, F^{31}[s]\, F^{01}{}_3[s] - \frac{2}{9} q N_1[s]\, \kappa_1[s]\, \kappa_2[s]\, F^{02}{}_1[s] + \frac{2}{9} q N_2[s]\, (\kappa_1)'[s]\, F^{02}{}_1[s] +$$

$$\frac{4}{3} c \pi \epsilon_0 N_2[s]\, F^{01}[s]\, F^{02}{}_1[s] - \frac{4}{3} c \pi \epsilon_0 N_1[s]\, F^{02}[s]\, F^{02}{}_1[s] - \frac{4}{3} c \pi \epsilon_0 N_0[s]\, F^{12}[s]\, F^{02}{}_1[s] + \frac{2}{9} q N_2[s]\, \kappa_1[s]\, \kappa_2[s]\, F^{02}{}_2[s] +$$

$$\frac{2}{9} q N_1[s]\, (\kappa_1)'[s]\, F^{02}{}_2[s] + \frac{4}{3} c \pi \epsilon_0 N_1[s]\, F^{01}[s]\, F^{02}{}_2[s] + \frac{4}{3} c \pi \epsilon_0 N_2[s]\, F^{02}[s]\, F^{02}{}_2[s] + \frac{4}{3} c \pi \epsilon_0 N_3[s]\, F^{03}[s]\, F^{02}{}_2[s] -$$

$$\frac{2}{9} q N_3[s]\, \kappa_1[s]\, \kappa_2[s]\, F^{02}{}_3[s] - \frac{4}{3} c \pi \epsilon_0 N_3[s]\, F^{02}[s]\, F^{02}{}_3[s] + \frac{4}{3} c \pi \epsilon_0 N_2[s]\, F^{03}[s]\, F^{02}{}_3[s] + \frac{4}{3} c \pi \epsilon_0 N_0[s]\, F^{23}[s]\, F^{02}{}_3[s] +$$

$$\frac{2}{9} q N_3[s]\, (\kappa_1)'[s]\, F^{03}{}_1[s] + \frac{4}{3} c \pi \epsilon_0 N_3[s]\, F^{01}[s]\, F^{03}{}_1[s] - \frac{4}{3} c \pi \epsilon_0 N_1[s]\, F^{03}[s]\, F^{03}{}_1[s] + \frac{4}{3} c \pi \epsilon_0 N_0[s]\, F^{31}[s]\, F^{03}{}_1[s] +$$

$$\frac{2}{9} q N_3[s]\, \kappa_1[s]\, \kappa_2[s]\, F^{03}{}_2[s] + \frac{4}{3} c \pi \epsilon_0 N_3[s]\, F^{02}[s]\, F^{03}{}_2[s] - \frac{4}{3} c \pi \epsilon_0 N_2[s]\, F^{03}[s]\, F^{03}{}_2[s] - \frac{4}{3} c \pi \epsilon_0 N_0[s]\, F^{23}[s]\, F^{03}{}_2[s] +$$

$$\frac{2}{9} q N_2[s]\, \kappa_1[s]\, \kappa_2[s]\, F^{03}{}_3[s] + \frac{2}{9} q N_1[s]\, (\kappa_1)'[s]\, F^{03}{}_3[s] + \frac{4}{3} c \pi \epsilon_0 N_1[s]\, F^{01}[s]\, F^{03}{}_3[s] + \frac{4}{3} c \pi \epsilon_0 N_2[s]\, F^{02}[s]\, F^{03}{}_3[s] +$$

$$\frac{4}{3} c \pi \epsilon_0 N_3[s]\, F^{03}[s]\, F^{03}{}_3[s] - \frac{2}{9} q N_0[s]\, \kappa_1[s]\, \kappa_2[s]\, F^{12}{}_1[s] - \frac{4}{3} c \pi \epsilon_0 N_0[s]\, F^{02}[s]\, F^{12}{}_1[s] - \frac{4}{3} c \pi \epsilon_0 N_1[s]\, F^{12}[s]\, F^{12}{}_1[s] +$$

$$\frac{4}{3} c \pi \epsilon_0 N_3[s]\, F^{23}[s]\, F^{12}{}_1[s] + \frac{2}{9} q N_0[s]\, (\kappa_1)'[s]\, F^{12}{}_2[s] + \frac{4}{3} c \pi \epsilon_0 N_0[s]\, F^{01}[s]\, F^{12}{}_2[s] - \frac{4}{3} c \pi \epsilon_0 N_2[s]\, F^{12}[s]\, F^{12}{}_2[s] +$$

$$\frac{4}{3} c \pi \epsilon_0 N_3[s] F^{31}[s] F^{12}{}_2[s] + \frac{4}{3} c \pi \epsilon_0 N_3[s] F^{12}[s] F^{12}{}_3[s] + \frac{4}{3} c \pi \epsilon_0 N_1[s] F^{23}[s] F^{12}{}_3[s] + \frac{4}{3} c \pi \epsilon_0 N_2[s] F^{31}[s] F^{12}{}_3[s] +$$

$$\frac{4}{3} c \pi \epsilon_0 N_3[s] F^{12}[s] F^{23}{}_1[s] + \frac{4}{3} c \pi \epsilon_0 N_1[s] F^{23}[s] F^{23}{}_1[s] + \frac{4}{3} c \pi \epsilon_0 N_2[s] F^{31}[s] F^{23}{}_1[s] - \frac{4}{3} c \pi \epsilon_0 N_0[s] F^{03}[s] F^{23}{}_2[s] -$$

$$\frac{4}{3} c \pi \epsilon_0 N_2[s] F^{23}[s] F^{23}{}_2[s] + \frac{4}{3} c \pi \epsilon_0 N_1[s] F^{31}[s] F^{23}{}_2[s] + \frac{2}{9} q N_0[s] \kappa_1[s] \kappa_2[s] F^{23}{}_3[s] + \frac{4}{3} c \pi \epsilon_0 N_0[s] F^{02}[s] F^{23}{}_3[s] +$$

$$\frac{4}{3} c \pi \epsilon_0 N_1[s] F^{12}[s] F^{23}{}_3[s] - \frac{4}{3} c \pi \epsilon_0 N_3[s] F^{23}[s] F^{23}{}_3[s] + \frac{4}{3} c \pi \epsilon_0 N_0[s] F^{03}[s] F^{31}{}_1[s] + \frac{4}{3} c \pi \epsilon_0 N_2[s] F^{23}[s] F^{31}{}_1[s] -$$

$$\frac{4}{3} c \pi \epsilon_0 N_1[s] F^{31}[s] F^{31}{}_1[s] + \frac{4}{3} c \pi \epsilon_0 N_3[s] F^{12}[s] F^{31}{}_2[s] + \frac{4}{3} c \pi \epsilon_0 N_1[s] F^{23}[s] F^{31}{}_2[s] + \frac{4}{3} c \pi \epsilon_0 N_2[s] F^{31}[s] F^{31}{}_2[s] -$$

$$\left.\frac{2}{9} q N_0[s] (\kappa_1)'[s] F^{31}{}_3[s] - \frac{4}{3} c \pi \epsilon_0 N_0[s] F^{01}[s] F^{31}{}_3[s] + \frac{4}{3} c \pi \epsilon_0 N_2[s] F^{12}[s] F^{31}{}_3[s] - \frac{4}{3} c \pi \epsilon_0 N_3[s] F^{31}[s] F^{31}{}_3[s]\right) r^3 + O[r]^4$$

dPdr[s, r]

Series expression for dP_{cap}/dr

$$\frac{q^2 N_0[s]}{8 c \pi \epsilon_0 r^2} + \left(\frac{q^2 N_0[s] \kappa_1[s]^2}{16 c \pi \epsilon_0} - \frac{q^2 N_2[s] \kappa_1[s] \kappa_2[s]}{12 c \pi \epsilon_0} - \frac{q^2 N_1[s] (\kappa_1)'[s]}{12 c \pi \epsilon_0}\right) +$$

$$\left(-\frac{q^2 N_1[s] \kappa_1[s] \kappa_2[s]^2}{18 c \pi \epsilon_0} + \frac{q^2 N_3[s] \kappa_1[s] \kappa_2[s] \kappa_3[s]}{18 c \pi \epsilon_0} - \frac{q^2 N_0[s] \kappa_1[s] (\kappa_1)'[s]}{9 c \pi \epsilon_0} + \frac{q^2 N_2[s] \kappa_2[s] (\kappa_1)'[s]}{9 c \pi \epsilon_0} + \frac{q^2 N_2[s] \kappa_1[s] (\kappa_2)'[s]}{18 c \pi \epsilon_0} +\right.$$

$$\frac{q^2 N_1[s] (\kappa_1)''[s]}{18 c \pi \epsilon_0} - \frac{2}{3} q N_0[s] \kappa_1[s] F^{01}[s] + \frac{2}{3} q N_2[s] \kappa_1[s] F^{12}[s] - \frac{2}{3} q N_3[s] \kappa_1[s] F^{31}[s] + \frac{1}{3} q N_0[s] F^{01}{}_1[s] + \frac{1}{3} q N_0[s] F^{02}{}_2[s] +$$

$$\left.\frac{1}{3} q N_0[s] F^{03}{}_3[s] - \frac{1}{3} q N_2[s] F^{12}{}_1[s] + \frac{1}{3} q N_1[s] F^{12}{}_2[s] - \frac{1}{3} q N_3[s] F^{23}{}_2[s] + \frac{1}{3} q N_2[s] F^{23}{}_3[s] + \frac{1}{3} q N_3[s] F^{31}{}_1[s] - \frac{1}{3} q N_1[s] F^{31}{}_3[s]\right) r +$$

$$\left(\frac{3 q^2 N_0[s] \kappa_1[s]^4}{128 c \pi \epsilon_0} - \frac{71 q^2 N_2[s] \kappa_1[s]^3 \kappa_2[s]}{480 c \pi \epsilon_0} + \frac{7 q^2 N_0[s] \kappa_1[s]^2 \kappa_2[s]^2}{288 c \pi \epsilon_0} + \frac{q^2 N_2[s] \kappa_1[s] \kappa_2[s]^3}{48 c \pi \epsilon_0} + \frac{q^2 N_2[s] \kappa_1[s] \kappa_2[s] \kappa_3[s]^2}{48 c \pi \epsilon_0} -\right.$$

$$\frac{3 q^2 N_1[s]\, \kappa_1[s]^2\, (\kappa_1)'[s]}{80\, c\, \pi\, \epsilon_0} + \frac{q^2 N_1[s]\, \kappa_2[s]^2\, (\kappa_1)'[s]}{16\, c\, \pi\, \epsilon_0} - \frac{q^2 N_3[s]\, \kappa_2[s]\, \kappa_3[s]\, (\kappa_1)'[s]}{16\, c\, \pi\, \epsilon_0} + \frac{11 q^2 N_0[s]\, (\kappa_1)'[s]^2}{144\, c\, \pi\, \epsilon_0} +$$

$$\frac{q^2 N_1[s]\, \kappa_1[s]\, \kappa_2[s]\, (\kappa_2)'[s]}{16\, c\, \pi\, \epsilon_0} - \frac{q^2 N_3[s]\, \kappa_1[s]\, \kappa_3[s]\, (\kappa_2)'[s]}{24\, c\, \pi\, \epsilon_0} - \frac{q^2 N_2[s]\, (\kappa_1)'[s]\, (\kappa_2)'[s]}{16\, c\, \pi\, \epsilon_0} - \frac{q^2 N_3[s]\, \kappa_1[s]\, \kappa_2[s]\, (\kappa_3)'[s]}{48\, c\, \pi\, \epsilon_0} +$$

$$\frac{5 q^2 N_0[s]\, \kappa_1[s]\, (\kappa_1)''[s]}{96\, c\, \pi\, \epsilon_0} - \frac{q^2 N_2[s]\, \kappa_2[s]\, (\kappa_1)''[s]}{16\, c\, \pi\, \epsilon_0} - \frac{q^2 N_2[s]\, \kappa_1[s]\, (\kappa_2)''[s]}{48\, c\, \pi\, \epsilon_0} - \frac{q^2 N_1[s]\, (\kappa_1)^{(3)}[s]}{48\, c\, \pi\, \epsilon_0} + \frac{2}{3} q\, N_0[s]\, (\kappa_1)'[s]\, F^{01}[s] +$$

$$2\, c\, \pi\, \epsilon_0\, N_0[s]\, F^{01}[s]^2 + \frac{2}{3} q\, N_0[s]\, \kappa_1[s]\, \kappa_2[s]\, F^{02}[s] + 2\, c\, \pi\, \epsilon_0\, N_0[s]\, F^{02}[s]^2 + 2\, c\, \pi\, \epsilon_0\, N_0[s]\, F^{03}[s]^2 + \frac{2}{3} q\, N_1[s]\, \kappa_1[s]\, \kappa_2[s]\, F^{12}[s] -$$

$$\frac{2}{3} q\, N_2[s]\, (\kappa_1)'[s]\, F^{12}[s] - 4\, c\, \pi\, \epsilon_0\, N_2[s]\, F^{01}[s]\, F^{12}[s] + 4\, c\, \pi\, \epsilon_0\, N_1[s]\, F^{02}[s]\, F^{12}[s] + 2\, c\, \pi\, \epsilon_0\, N_0[s]\, F^{12}[s]^2 -$$

$$\frac{2}{3} q\, N_3[s]\, \kappa_1[s]\, \kappa_2[s]\, F^{23}[s] - 4\, c\, \pi\, \epsilon_0\, N_3[s]\, F^{02}[s]\, F^{23}[s] + 4\, c\, \pi\, \epsilon_0\, N_2[s]\, F^{03}[s]\, F^{23}[s] + 2\, c\, \pi\, \epsilon_0\, N_0[s]\, F^{23}[s]^2 +$$

$$\left.\frac{2}{3} q\, N_3[s]\, (\kappa_1)'[s]\, F^{31}[s] + 4\, c\, \pi\, \epsilon_0\, N_3[s]\, F^{01}[s]\, F^{31}[s] - 4\, c\, \pi\, \epsilon_0\, N_1[s]\, F^{03}[s]\, F^{31}[s] + 2\, c\, \pi\, \epsilon_0\, N_0[s]\, F^{31}[s]^2\right) r^2 +$$

$$\left(-\frac{8 q^2 N_1[s]\, \kappa_1[s]^3\, \kappa_2[s]^2}{45\, c\, \pi\, \epsilon_0} + \frac{q^2 N_1[s]\, \kappa_1[s]\, \kappa_2[s]^4}{180\, c\, \pi\, \epsilon_0} + \frac{5 q^2 N_3[s]\, \kappa_1[s]^3\, \kappa_2[s]\, \kappa_3[s]}{36\, c\, \pi\, \epsilon_0} - \frac{q^2 N_3[s]\, \kappa_1[s]\, \kappa_2[s]^3\, \kappa_3[s]}{180\, c\, \pi\, \epsilon_0} +\right.$$

$$\frac{q^2 N_1[s]\, \kappa_1[s]\, \kappa_2[s]^2\, \kappa_3[s]^2}{180\, c\, \pi\, \epsilon_0} - \frac{q^2 N_3[s]\, \kappa_1[s]\, \kappa_2[s]\, \kappa_3[s]^3}{180\, c\, \pi\, \epsilon_0} - \frac{2 q^2 N_0[s]\, \kappa_1[s]^3\, (\kappa_1)'[s]}{15\, c\, \pi\, \epsilon_0} + \frac{37 q^2 N_2[s]\, \kappa_1[s]^2\, \kappa_2[s]\, (\kappa_1)'[s]}{90\, c\, \pi\, \epsilon_0} -$$

$$\frac{q^2 N_0[s]\, \kappa_1[s]\, \kappa_2[s]^2\, (\kappa_1)'[s]}{30\, c\, \pi\, \epsilon_0} - \frac{q^2 N_2[s]\, \kappa_2[s]^3\, (\kappa_1)'[s]}{45\, c\, \pi\, \epsilon_0} - \frac{q^2 N_2[s]\, \kappa_2[s]\, \kappa_3[s]^2\, (\kappa_1)'[s]}{45\, c\, \pi\, \epsilon_0} + \frac{5 q^2 N_2[s]\, \kappa_1[s]^3\, (\kappa_2)'[s]}{36\, c\, \pi\, \epsilon_0} -$$

$$\frac{q^2 N_0[s]\, \kappa_1[s]^2\, \kappa_2[s]\, (\kappa_2)'[s]}{30\, c\, \pi\, \epsilon_0} - \frac{q^2 N_2[s]\, \kappa_1[s]\, \kappa_2[s]^2\, (\kappa_2)'[s]}{30\, c\, \pi\, \epsilon_0} - \frac{q^2 N_2[s]\, \kappa_1[s]\, \kappa_3[s]^2\, (\kappa_2)'[s]}{60\, c\, \pi\, \epsilon_0} - \frac{q^2 N_1[s]\, \kappa_2[s]\, (\kappa_1)'[s]\, (\kappa_2)'[s]}{15\, c\, \pi\, \epsilon_0} +$$

$$\frac{2 q^2 N_3[s]\, \kappa_3[s]\, (\kappa_1)'[s]\, (\kappa_2)'[s]}{45\, c\, \pi\, \epsilon_0} - \frac{q^2 N_1[s]\, \kappa_1[s]\, (\kappa_2)'[s]^2}{60\, c\, \pi\, \epsilon_0} - \frac{q^2 N_2[s]\, \kappa_1[s]\, \kappa_2[s]\, \kappa_3[s]\, (\kappa_3)'[s]}{60\, c\, \pi\, \epsilon_0} + \frac{q^2 N_3[s]\, \kappa_2[s]\, (\kappa_1)'[s]\, (\kappa_3)'[s]}{45\, c\, \pi\, \epsilon_0} +$$

$$\frac{q^2\,N_3[s]\,\kappa_1[s]\,(\kappa_2)'[s]\,(\kappa_3)'[s]}{60\,c\,\pi\,\epsilon_0}+\frac{2\,q^2\,N_1[s]\,\kappa_1[s]^2\,(\kappa_1)''[s]}{45\,c\,\pi\,\epsilon_0}-\frac{q^2\,N_1[s]\,\kappa_2[s]^2\,(\kappa_1)''[s]}{30\,c\,\pi\,\epsilon_0}+\frac{q^2\,N_3[s]\,\kappa_2[s]\,\kappa_3[s]\,(\kappa_1)''[s]}{30\,c\,\pi\,\epsilon_0}-$$

$$\frac{q^2\,N_0[s]\,(\kappa_1)'[s]\,(\kappa_1)''[s]}{12\,c\,\pi\,\epsilon_0}+\frac{q^2\,N_2[s]\,(\kappa_2)'[s]\,(\kappa_1)''[s]}{30\,c\,\pi\,\epsilon_0}-\frac{q^2\,N_1[s]\,\kappa_1[s]\,\kappa_2[s]\,(\kappa_2)''[s]}{45\,c\,\pi\,\epsilon_0}+\frac{q^2\,N_3[s]\,\kappa_1[s]\,\kappa_3[s]\,(\kappa_2)''[s]}{60\,c\,\pi\,\epsilon_0}+$$

$$\frac{q^2\,N_2[s]\,(\kappa_1)'[s]\,(\kappa_2)''[s]}{45\,c\,\pi\,\epsilon_0}+\frac{q^2\,N_3[s]\,\kappa_1[s]\,\kappa_2[s]\,(\kappa_3)''[s]}{180\,c\,\pi\,\epsilon_0}-\frac{q^2\,N_0[s]\,\kappa_1[s]\,(\kappa_1)^{(3)}[s]}{60\,c\,\pi\,\epsilon_0}+\frac{q^2\,N_2[s]\,\kappa_2[s]\,(\kappa_1)^{(3)}[s]}{45\,c\,\pi\,\epsilon_0}+$$

$$\frac{q^2\,N_2[s]\,\kappa_1[s]\,(\kappa_2)^{(3)}[s]}{180\,c\,\pi\,\epsilon_0}+\frac{q^2\,N_1[s]\,(\kappa_1)^{(4)}[s]}{180\,c\,\pi\,\epsilon_0}+\frac{1}{3}\,q\,N_2[s]\,\kappa_1[s]^2\,\kappa_2[s]\,F^{01}[s]+\frac{1}{3}\,q\,N_0[s]\,\kappa_1[s]\,\kappa_2[s]^2\,F^{01}[s]-\frac{1}{3}\,q\,N_0[s]\,(\kappa_1)''[s]\,F^{01}[s]-$$

$$\frac{1}{3}\,q\,N_1[s]\,\kappa_1[s]^2\,\kappa_2[s]\,F^{02}[s]-\frac{2}{3}\,q\,N_0[s]\,\kappa_2[s]\,(\kappa_1)'[s]\,F^{02}[s]-\frac{1}{3}\,q\,N_0[s]\,\kappa_1[s]\,(\kappa_2)'[s]\,F^{02}[s]-\frac{1}{3}\,q\,N_0[s]\,\kappa_1[s]\,\kappa_2[s]\,\kappa_3[s]\,F^{03}[s]-$$

$$\frac{1}{3}\,q\,N_0[s]\,\kappa_1[s]^2\,\kappa_2[s]\,F^{12}[s]-\frac{1}{3}\,q\,N_2[s]\,\kappa_1[s]\,\kappa_2[s]^2\,F^{12}[s]-\frac{2}{3}\,q\,N_1[s]\,\kappa_2[s]\,(\kappa_1)'[s]\,F^{12}[s]-\frac{1}{3}\,q\,N_1[s]\,\kappa_1[s]\,(\kappa_2)'[s]\,F^{12}[s]+$$

$$\frac{1}{3}\,q\,N_2[s]\,(\kappa_1)''[s]\,F^{12}[s]-\frac{1}{3}\,q\,N_2[s]\,\kappa_1[s]\,\kappa_2[s]\,\kappa_3[s]\,F^{23}[s]+\frac{2}{3}\,q\,N_3[s]\,\kappa_2[s]\,(\kappa_1)'[s]\,F^{23}[s]+\frac{1}{3}\,q\,N_3[s]\,\kappa_1[s]\,(\kappa_2)'[s]\,F^{23}[s]+$$

$$\frac{1}{3}\,q\,N_3[s]\,\kappa_1[s]\,\kappa_2[s]^2\,F^{31}[s]+\frac{1}{3}\,q\,N_1[s]\,\kappa_1[s]\,\kappa_2[s]\,\kappa_3[s]\,F^{31}[s]-\frac{1}{3}\,q\,N_3[s]\,(\kappa_1)''[s]\,F^{31}[s]+\frac{1}{5}\,q\,N_0[s]\,\kappa_1[s]^2\,F^{01}{}_1[s]-$$

$$\frac{1}{6}\,q\,N_2[s]\,\kappa_1[s]\,\kappa_2[s]\,F^{01}{}_1[s]+\frac{1}{6}\,q\,N_2[s]\,(\kappa_1)'[s]\,F^{01}{}_2[s]+\frac{1}{6}\,q\,N_3[s]\,(\kappa_1)'[s]\,F^{01}{}_3[s]+\frac{1}{6}\,q\,N_1[s]\,\kappa_1[s]\,\kappa_2[s]\,F^{02}{}_1[s]-$$

$$\frac{1}{10}\,q\,N_0[s]\,\kappa_1[s]^2\,F^{02}{}_2[s]-\frac{1}{6}\,q\,N_1[s]\,(\kappa_1)'[s]\,F^{02}{}_2[s]+\frac{1}{6}\,q\,N_3[s]\,\kappa_1[s]\,\kappa_2[s]\,F^{02}{}_3[s]-\frac{1}{10}\,q\,N_0[s]\,\kappa_1[s]^2\,F^{03}{}_3[s]-$$

$$\frac{1}{6}\,q\,N_2[s]\,\kappa_1[s]\,\kappa_2[s]\,F^{03}{}_3[s]-\frac{1}{6}\,q\,N_1[s]\,(\kappa_1)'[s]\,F^{03}{}_3[s]-\frac{1}{5}\,q\,N_2[s]\,\kappa_1[s]^2\,F^{12}{}_1[s]+\frac{1}{6}\,q\,N_0[s]\,\kappa_1[s]\,\kappa_2[s]\,F^{12}{}_1[s]-$$

$$\frac{1}{10}\,q\,N_1[s]\,\kappa_1[s]^2\,F^{12}{}_2[s]-\frac{1}{6}\,q\,N_0[s]\,(\kappa_1)'[s]\,F^{12}{}_2[s]+\frac{1}{10}\,q\,N_3[s]\,\kappa_1[s]^2\,F^{23}{}_2[s]-\frac{1}{10}\,q\,N_2[s]\,\kappa_1[s]^2\,F^{23}{}_3[s]-$$

$$\left.\frac{1}{6}\,q\,N_0[s]\,\kappa_1[s]\,\kappa_2[s]\,F^{23}{}_3[s]+\frac{1}{5}\,q\,N_3[s]\,\kappa_1[s]^2\,F^{31}{}_1[s]+\frac{1}{10}\,q\,N_1[s]\,\kappa_1[s]^2\,F^{31}{}_3[s]+\frac{1}{6}\,q\,N_0[s]\,(\kappa_1)'[s]\,F^{31}{}_3[s]\right)\,r^3+O[r]^4$$

```
dPds2nd[s_, r_] = D[p[s] + Integrate[dPdr[s, r], r], s];
```

```
zeroBalanceEqn = ExpandAll[ dPds[s, r] - dPds2nd[s, r]  /. MaxwellSet1]
```

$$\left(\frac{q^2 N_2[s]\,\kappa_1[s]\,\kappa_2[s]}{6\,c\,\pi\,\epsilon_0} - p'[s] + \frac{q^2 N_1[s]\,(\kappa_1)'[s]}{6\,c\,\pi\,\epsilon_0} + q N_1[s]\,F^{01}[s] + q N_2[s]\,F^{02}[s] + q N_3[s]\,F^{03}[s]\right) + O[r]^4 \qquad *$$

The simplification rules used above are that the background field F satisfy the source-free Maxwell equations along the particle worldline:

```
MaxwellSet1
```

$$\{F^{01}{}_0[s] \to F^{12}{}_2[s] - F^{31}{}_3[s],\ F^{02}{}_0[s] \to -F^{12}{}_1[s] + F^{23}{}_3[s],\ F^{03}{}_0[s] \to -F^{23}{}_2[s] + F^{31}{}_1[s],\ F^{23}{}_0[s] \to F^{02}{}_3[s] - F^{03}{}_2[s],$$
$$F^{31}{}_0[s] \to -F^{01}{}_3[s] + F^{03}{}_1[s],\ F^{12}{}_0[s] \to F^{01}{}_2[s] - F^{02}{}_1[s],\ F^{12}{}_3[s] \to -F^{23}{}_1[s] - F^{31}{}_2[s],\ F^{03}{}_3[s] \to -F^{01}{}_1[s] - F^{02}{}_2[s]\}$$

The balance equation is $\dot p^\alpha = q\,F^\alpha{}_\beta\,\dot z^\beta + \dfrac{q^2}{6\pi\epsilon_0 c} D_F\,\ddot z^\alpha$ *

where D_F is the Fermi derivative along the particle worldline,

$$D_F \ddot z^\alpha = (\delta^\alpha_\beta + \dot z^\alpha \dot z_\beta)\,\dddot z^\beta$$
$$= \dddot z^\alpha - (\kappa_1)^2 \dot z^\alpha$$
$$= N_1 \dot\kappa_1 + N_2 \kappa_1 \kappa_2 .$$

The balance equation becomes an equation of motion only after specifying how the momentum should be defined. The usual 1st order Lagrangian expression is $p = cm\dot z$, ie. mass x 4-velocity. With this, * becomes the L-D eqn.