The Conformal Structure of the FRW Space-times

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Acknowledgments

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Introduction

We want to understand the geometrical structure of cosmological origins and futures.

Quiescent Cosmology

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The universe is isotropic on a large scale because we are at a sufficiently early stage in its evolution.

Opposite view as compared to Chaotic Cosmology

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Chaotic Cosmology $\implies$ the early universe was very disordered.

The universe is isotropic on a large scale because we are at a sufficiently late stage in its evolution.

We will concern ourselves with Quiescent Cosmology today.

An Isotropic Start to the Universe...

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- Achieved by Goode and Wainwright (1985):
  - Was focused on the past of the universe (Isotropic Singularity),
  - Influenced people to consider the end of the universe,
  - Laid foundations for a possible useable framework of Quiescent Cosmology via Conformal Cosmology...

Conformal Cosmology

Conformal cosmology is useful:

- Preserves null cone structure,
- "Smooths" out singular behaviour and
- Is very general

$$g_{ab} = \Omega^2 \tilde{g}_{ab} \quad (1)$$

Conformal Definitions

Scott and Höhn have made substantial progress in coming up with a frame work for Quiescent Cosmology using Conformal Cosmology[1].

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- Future Isotropic Universe
- Anisotropic Future Endless Universe

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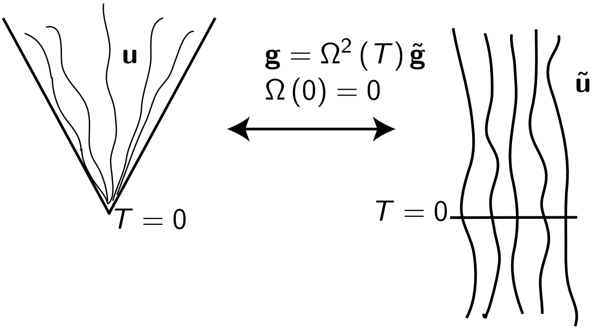
The Isotropic Past Singularity[1]

Physical space-time
$(\mathcal{M}, \mathbf{g})$

Unphysical space-time
$(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$



The Isotropic Future Singularity[1]

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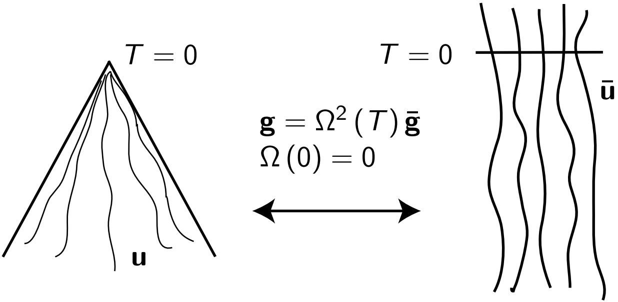

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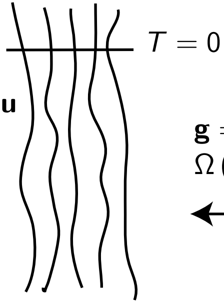
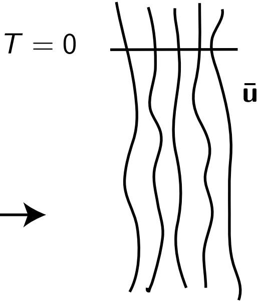


$$\mathbf{g} = \Omega^2(T)\,\bar{\mathbf{g}}$$
$$\Omega(0) = +\infty$$



The FRW Space-times

The Friedmann-Robertson-Walker (FRW) solution to the Einstein field equations describes space-times that

- are spherically symmetric,
- homogeneous,
- isotropic and
- are able to expand and/or contract.

The FRW Space-times

The FRW metric is given by

$$ds^2 = -dt^2 + a^2(t)d\sigma^2 \qquad (2)$$

where

$$d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta^2 d\phi^2 \qquad (3)$$

and $a(t)$ is known as the scale factor.

Scale factor $\Longrightarrow$ radius of curvature

The FRW space-times

The scale factor can be expressed as a power series around a point in time, t_0,

$$a(t) = \sum_{i=0} c_i |t - t_0|^{\eta_i} \tag{4}$$

Cosmological events in FRW space-times

Visser and Cattoén have made an extensive catalogue of cosmological events in FRW space-times[2].

[2] Matt Visser and Celine Cattoén, Necessary and sufficient conditions for big bangs, bounces, crunches, rips, sudden singularities and extremality events, *Classical and Quantum Gravity* Vol 22, 2009

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Cosmological Events in FRW Space-times

A FRW space-time is said to admit a Big Bang/Crunch type singularity if the scale factor goes to zero as $t \to {t_0}^{\pm}$.

It will be a Big Bang if we approach t_0 from above and a Big Crunch if we approach t_0 from below.

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A FRW space-time contains a Big Rip type singularity if the scale factor diverges as $t \longrightarrow t_0$.

Conformally Related FRW Space-times

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"Science is either physics or stamp collecting".

Ernest Rutherford. (Winner of the 1908 Nobel Prize in Chemistry)

A Conformal Relation for the FRW space-time

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Set

$$\Omega = (-T)^{\frac{\eta_0}{1-\eta_0}} \quad (7)$$
$$\text{where}(-T) = (t_0 - t)^{1-\eta_0} \quad (8)$$

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This means that

$$ds^2 = \Omega^2(T)\left(\frac{dT^2}{(1-\eta_0)^2} + {c_0}^2 d\sigma^2\right) \tag{9}$$

A Conformal Relation for the FRW Space-time

For $\eta_0 \in (0, 1)$ the conformally related FRW space-time admits an IFS.

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\begin{aligned}
\lim_{T\to 0^+} \Omega(T) &= (-T)^{\frac{\eta_0}{1-\eta_0}} \to 0 \\
\tilde{\mathbf{g}} &= \frac{dT^2}{(1-\eta_0)^2} + {c_0}^2 d\sigma^2 \\
\bar{\lambda} &\equiv \lim_{T\to 0^+} \bar{L}(T) = \frac{\Omega''}{\Omega}\left(\frac{\Omega}{\Omega'}\right)^2 = \frac{2\eta_0 - 1}{\eta_0}
\end{aligned}
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For $\eta_0 < 0$ it admits a FIU.

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A conformal Relation for the FRW Space-time

We propose that the Big Crunch is a generalisation of an IFS.

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We propose that the Big Rip is a subclass of a FIU.

Conclusions and Further Work

We have new structures for the singularities in the FRW space-times in terms of conformal definitions.

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Develop a new definition that takes into account sudden singularities and extremality events.

Conclusions and Further Work

We have new structures for the singularities in the FRW space-times in terms of conformal definitions.

It is easy to examine the behaviour of a conformally related space-time because it depends only on the conformal factor Ω.

Develop a new definition that takes into account sudden singularities and extremality events.

Further expand the technical aspects of Conformal Cosmology

Definition (Isotropic Past Singularity (IPS)[3])

A space-time, $(\mathcal{M}, \mathbf{g})$ admits an isotropic past singularity if there exists a space-time $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$, a smooth cosmic time function T defined on $\tilde{\mathcal{M}}$ and a conformal factor $\tilde{\Omega}(T)$ which satisfy

1. $\mathcal{M}$ is the open neighbourhood $T > 0$,
2. $\mathbf{g} = \tilde{\Omega}^2(T)\tilde{\mathbf{g}}$ on $\mathcal{M}$, with $\tilde{\mathbf{g}}$ regular (at least C^3 and non-degenerate) on an open neighbourhood of $T = 0$,
3. $\Omega(0) = 0$ and $\exists\ b > 0$ such that $\Omega \in C^0[0, b] \cap C^3(0, b]$ and $\Omega(0, b] > 0$,
4. $\lambda \equiv \lim_{T \to 0^+} L(T)$ exists, $\lambda \neq 1$, where $L \equiv \frac{\Omega''}{\Omega}\left(\frac{\Omega}{\Omega'}\right)^2$ and a prime denoted differentiation with respect to T.

[3] Scott, 1998

It turns out that to have true initial isotropy we must also constrain the fluid flow, as follows

Definition (Fluid Congruence of an IPS)

With any unit time-like congruence $\mathbf{u}$ in $\mathcal{M}$ we can associate a unit time-like congruence $\tilde{\mathbf{u}}$ in $\tilde{\mathcal{M}}$ such that

$$\tilde{\mathbf{u}} = \Omega \mathbf{u} \qquad \text{in } \mathcal{M}. \tag{10}$$

- If we can choose $\tilde{\mathbf{u}}$ to be regular (at least C^3) on an open neighbourhood of $T = 0$ in $\mathcal{M}$, we can say that $\mathbf{u}$ is regular at the IPS.
- If, in addition, $\tilde{\mathbf{u}}$ is orthogonal to $T = 0$, we say that $\mathbf{u}$ is orthogonal to the IPS.

Definition (Isotropic Future Singularity (IFS))

A space-time, $(\mathcal{M}, \mathbf{g})$ admits an isotropic future singularity if there exists a space-time $(\bar{\mathcal{M}}, \bar{\mathbf{g}})$, a smooth cosmic time function T defined on $\bar{\mathcal{M}}$ and a conformal factor $\Omega(T)$ which satisfy

1. $\mathcal{M}$ is the open neighbourhood $T < 0$,
2. $\mathbf{g} = \Omega^2(T)\bar{\mathbf{g}}$ on $\mathcal{M}$, with $\bar{\mathbf{g}}$ regular (at least C^2 and non-degenerate) on an open neighbourhood of $T = 0$,
3. $\Omega(0) = 0$ and $\exists\ c > 0$ such that $\Omega \in C^0[-c, 0] \cap C^2[-c, 0)$ and Ω is positive on $[-c, 0)$,
4. $\bar{\lambda} \equiv \lim\limits_{T \to 0^+} \bar{L}(T)$ exists, $\bar{\lambda} \neq 1$, where $\bar{L} \equiv \frac{\Omega''}{\Omega}\left(\frac{\Omega}{\Omega'}\right)^2$ and a prime denoted differentiation with respect to T.

Definition (Fluid Congruence of an IFS)

With any unit time-like congruence $\mathbf{u}$ in $\mathcal{M}$ we can associate a unit time-like congruence $\bar{\mathbf{u}}$ in $\bar{\mathcal{M}}$ such that

$$\bar{\mathbf{u}} = \Omega \mathbf{u} \qquad \text{in } \mathcal{M}. \tag{11}$$

- If we can choose $\bar{\mathbf{u}}$ to be regular (at least C^2) on an open neighbourhood of $T = 0$ in $\bar{\mathcal{M}}$, we can say that $\mathbf{u}$ is regular at the IFS.
- If, in addition, $\bar{\mathbf{u}}$ is orthogonal to $T = 0$, we say that $\mathbf{u}$ is orthogonal to the IFS.

Definition (Future Isotropic Universe (FIU))

A space-time, $(\mathcal{M}, \mathbf{g})$ admits an future isotropic universe if there exists a space-time $(\bar{\mathcal{M}}, \bar{\mathbf{g}})$, a smooth cosmic time function T defined on $\bar{\mathcal{M}}$ and a conformal factor $\Omega(T)$ which satisfy

1. $\lim_{T \to 0^-} \Omega(T) = +\infty$ and $\exists\ c > 0$ such that $\Omega \in C^2[-c, 0) \cap C^2[-c, 0)$ and Ω is strictly monotonically increasing and positive on $[-c, 0)$,
2. $\bar{\lambda}$ as defined above exists, $\bar{\lambda} \neq 1, 2$, and $\bar{L}$ is continuous on $[-c, 0)$ and
3. otherwise the conditions of the previous definitions are satisfied.