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Two new conditions for the occurrence of Krolak strong curvature singularities

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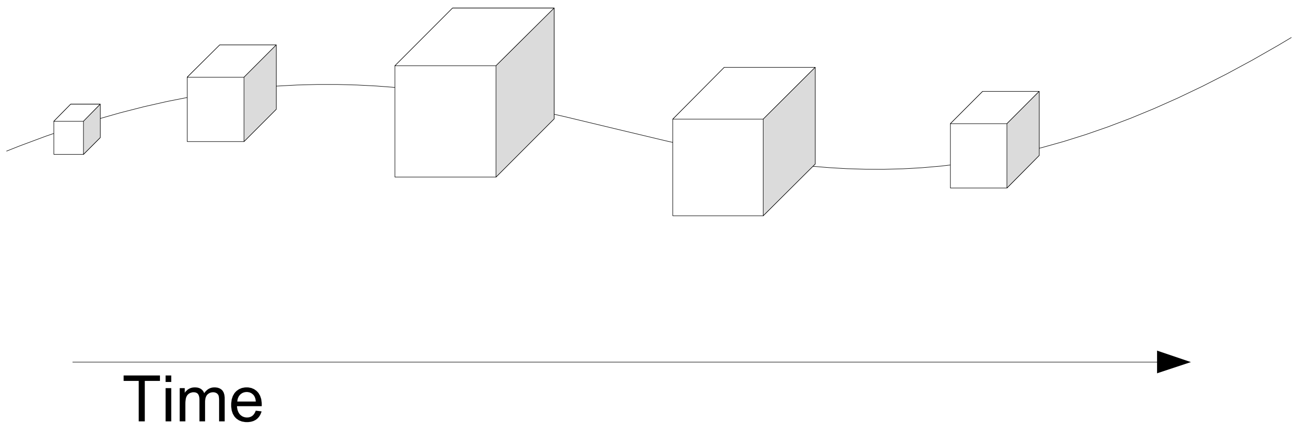
Ashley, M. J. S. L. and S. M. Scott, *Curvature singularities and abstract boundary singularity theorems for space-time.* Contem. Math. Vol. 337, pages 9-19.

Clarke, C.J.S. and A. Królak, *Conditions for the occurrence of strong curvature singularities.* J. Geom. Phys., 1985. **2**(2): p. 127-143.

Hawking, S.W., *Comments on cosmic censorship.* General Relativity and Gravitation, 1979. **10**: p. 1047 – 1049.

Why more conditions?



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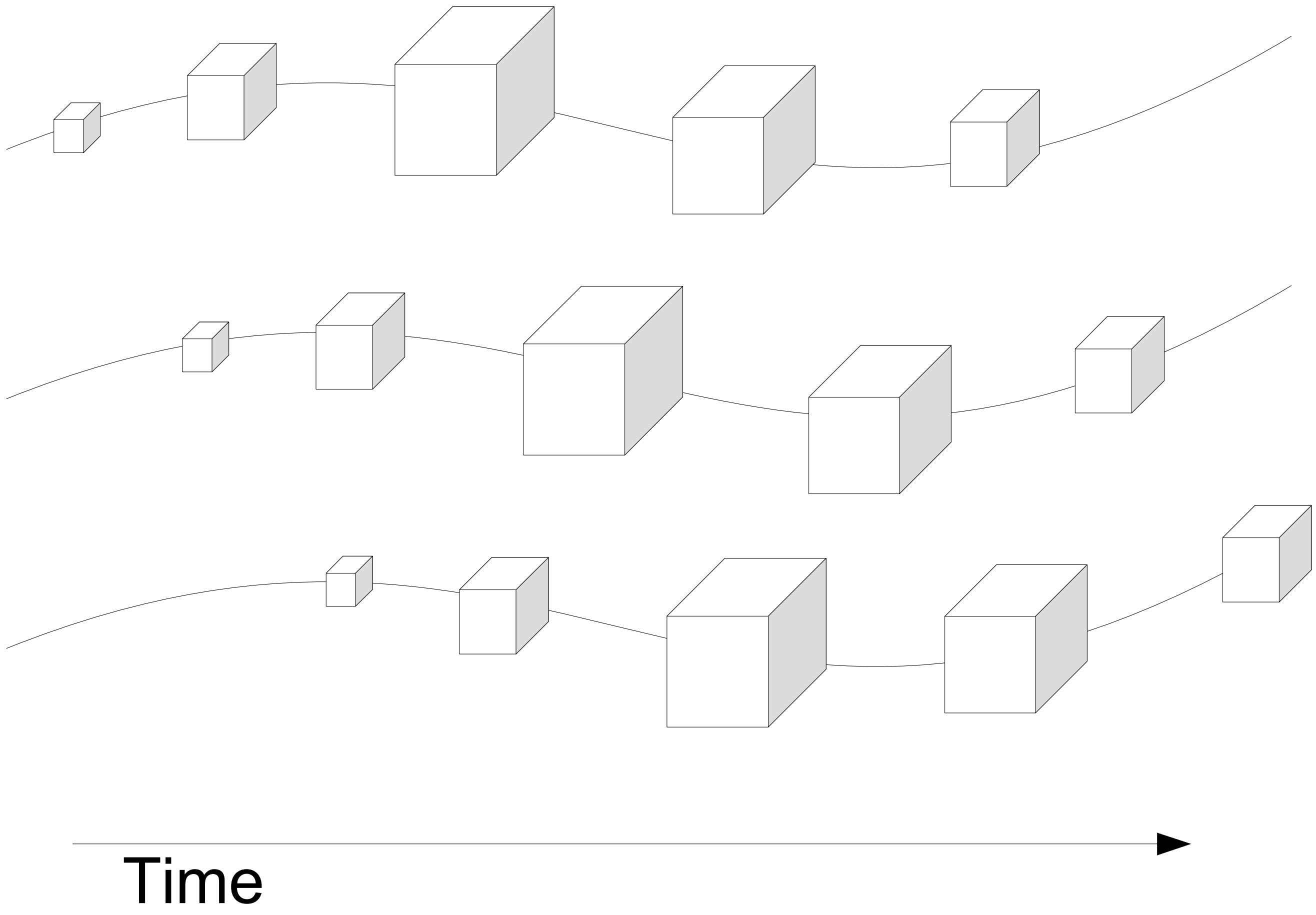
Why more conditions?



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Why more conditions?

The integral $\int \left| R^{i}{}_{0j0}(t) \right| dt$ does not converge along the geodesic

The integral $\int \left| R_{00}(t) \right| dt$ diverges to infinity

Clarke, C.J.S. and A. Królak, *Conditions for the occurrence of strong curvature singularities.* J. Geom. Phys., 1985. **2**(2): p. 127-143.

Jacobi tensors

Jacobi Tensor along a timelike geodesic $c:[0,a)\longrightarrow M$

$$J: T_{c'}^{\perp}M \to T_{c'}^{\perp}M,$$

$$\nabla_t \nabla_t J + R(J, c')c' = 0.$$

Beem, J.K., P.E. Ehrlich, and K.L. Easley, *Global Lorentzian Geometry*. Vol. 202. 1996: Marcel Dekker, Inc.

Divergence from a point

Geodesic divergence at $p = c(t_0)$,

$$\theta(t_0, t) = \operatorname{tr}\left(J^{-1}\nabla_t J\right) = \frac{1}{\det J}\frac{d}{dt}\det J$$

where,

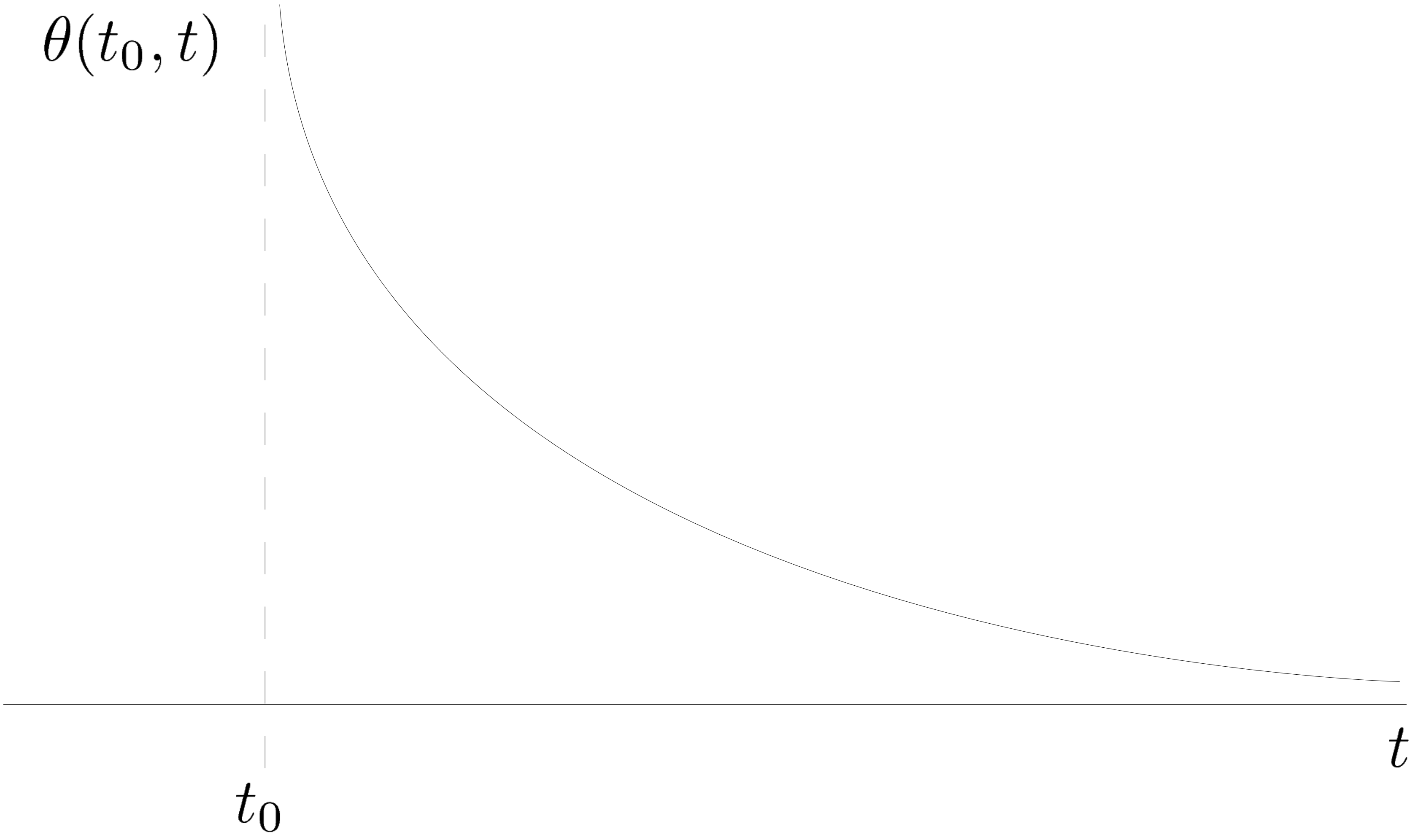
$$J(t_0) = 0$$

$$\nabla_t J\big|_{t_0} = I.$$

Beem, J.K., P.E. Ehrlich, and K.L. Easley, *Global Lorentzian Geometry*. Vol. 202. 1996: Marcel Dekker, Inc.

Illustration of divergence

Assuming the strong energy condition $R(c', c') \geq 0$,



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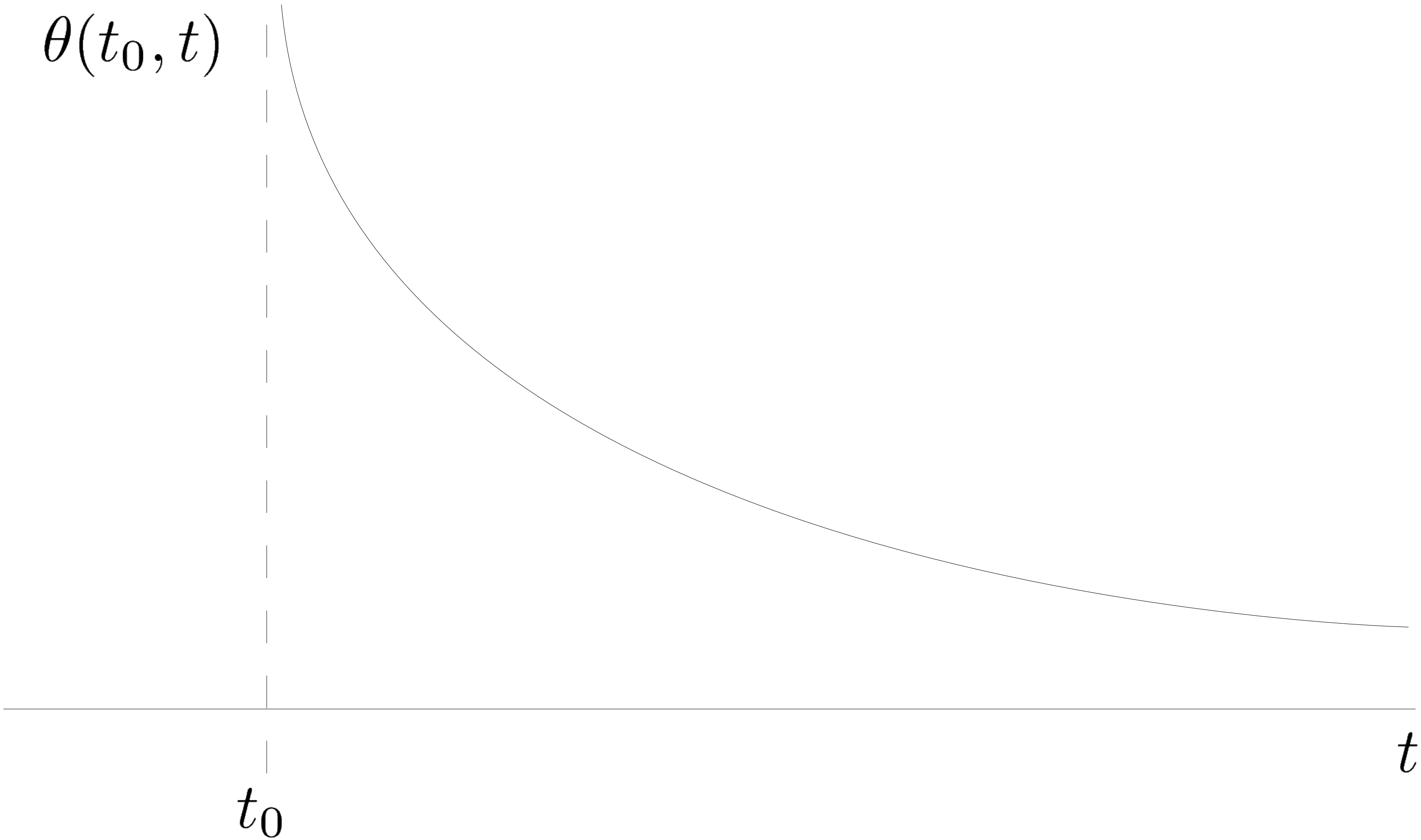

Illustration of divergence

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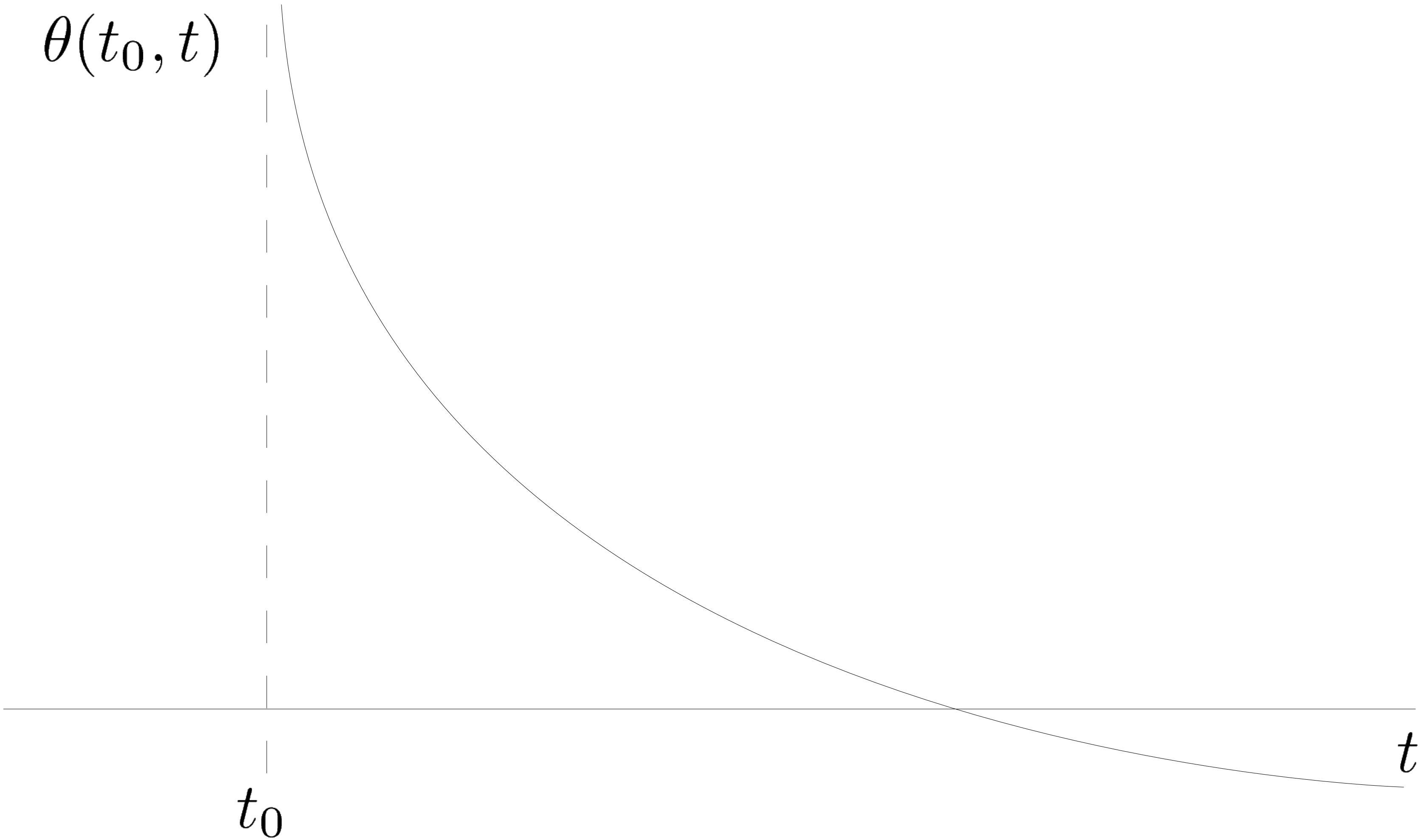
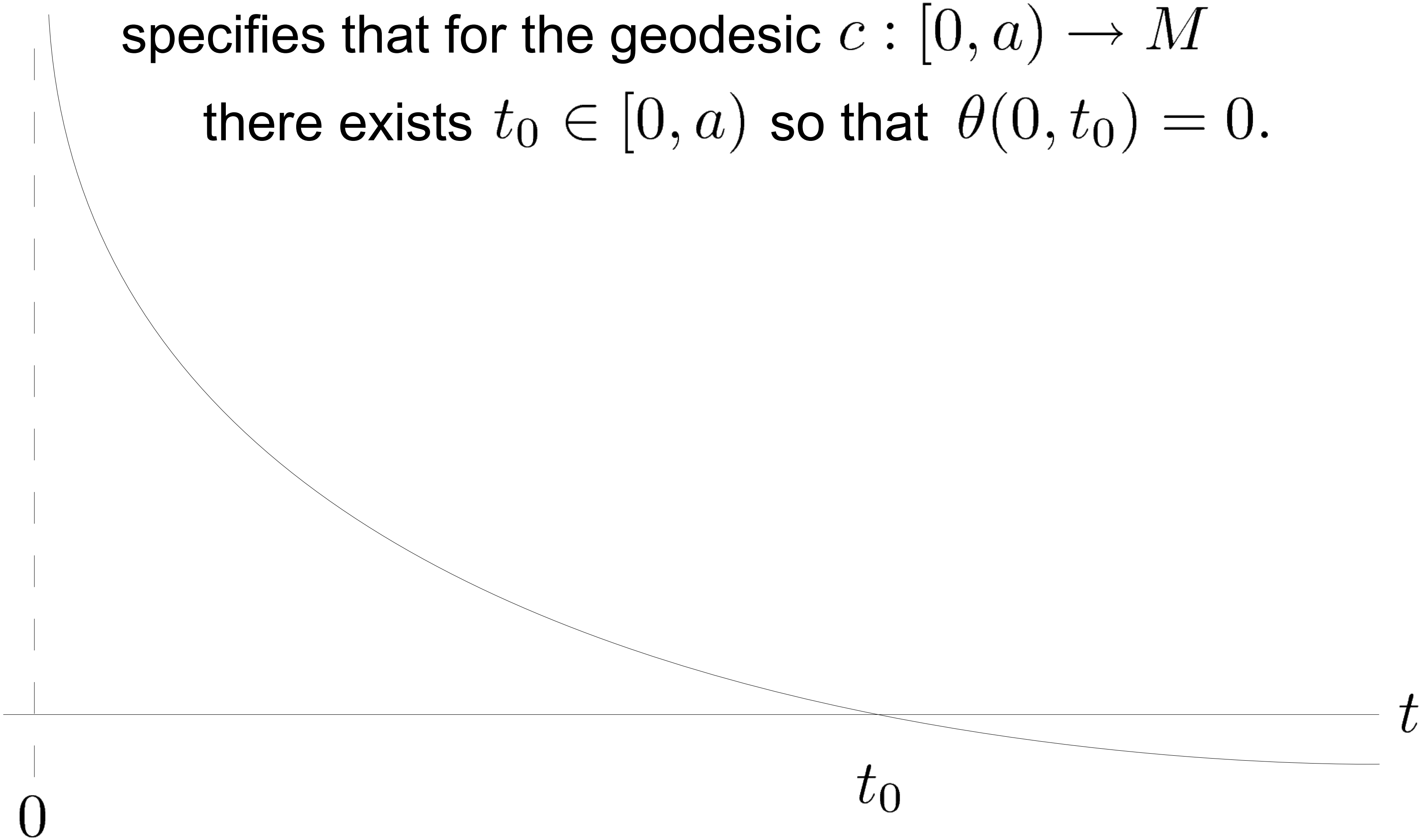


Point collapse condition

The point collapse condition of the singularity theorems specifies that for the geodesic $c : [0, a) \longrightarrow M$ there exists $t_0 \in [0, a)$ so that $\theta(0, t_0) = 0$.

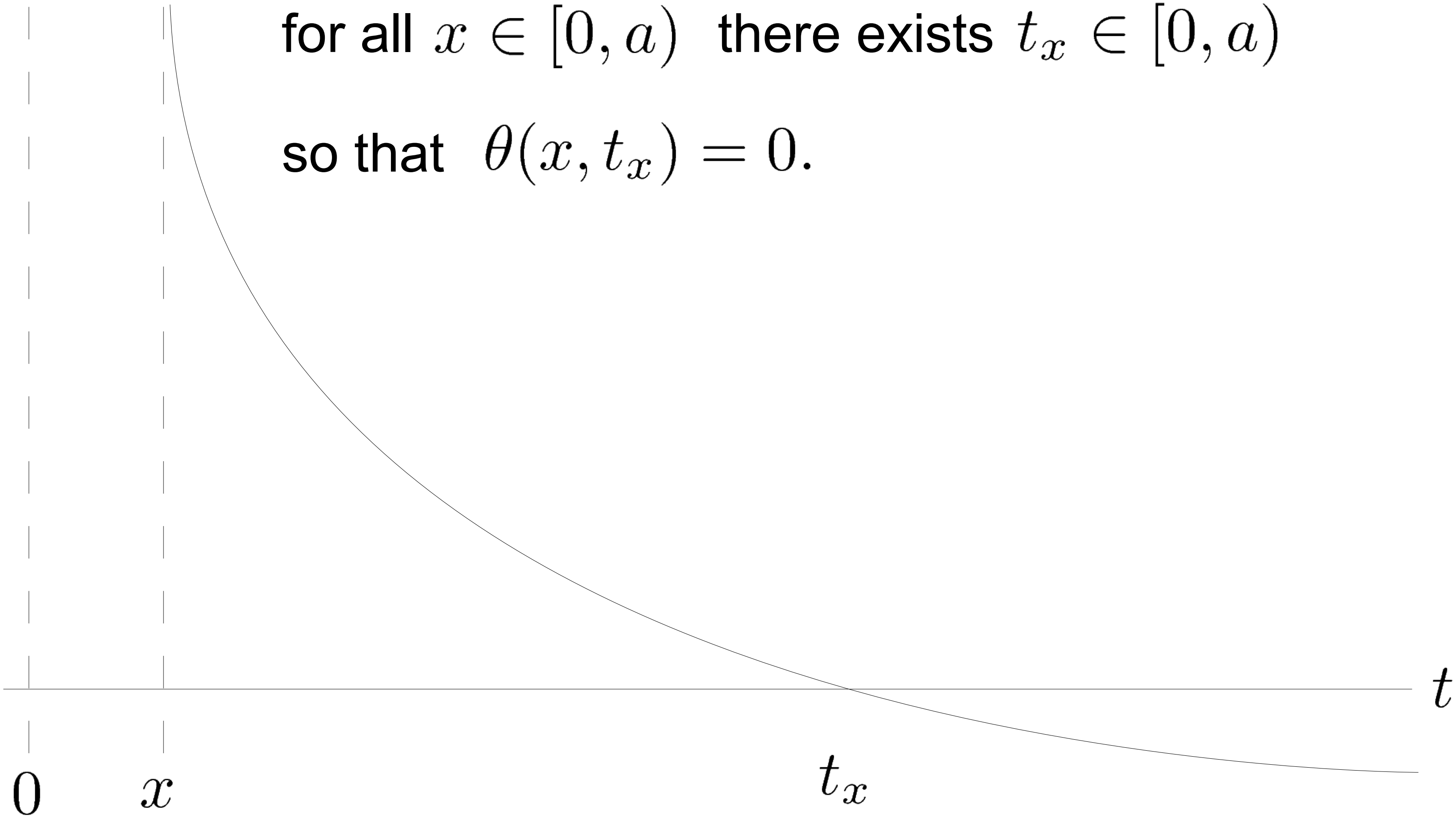


Krolak strong curvature condition

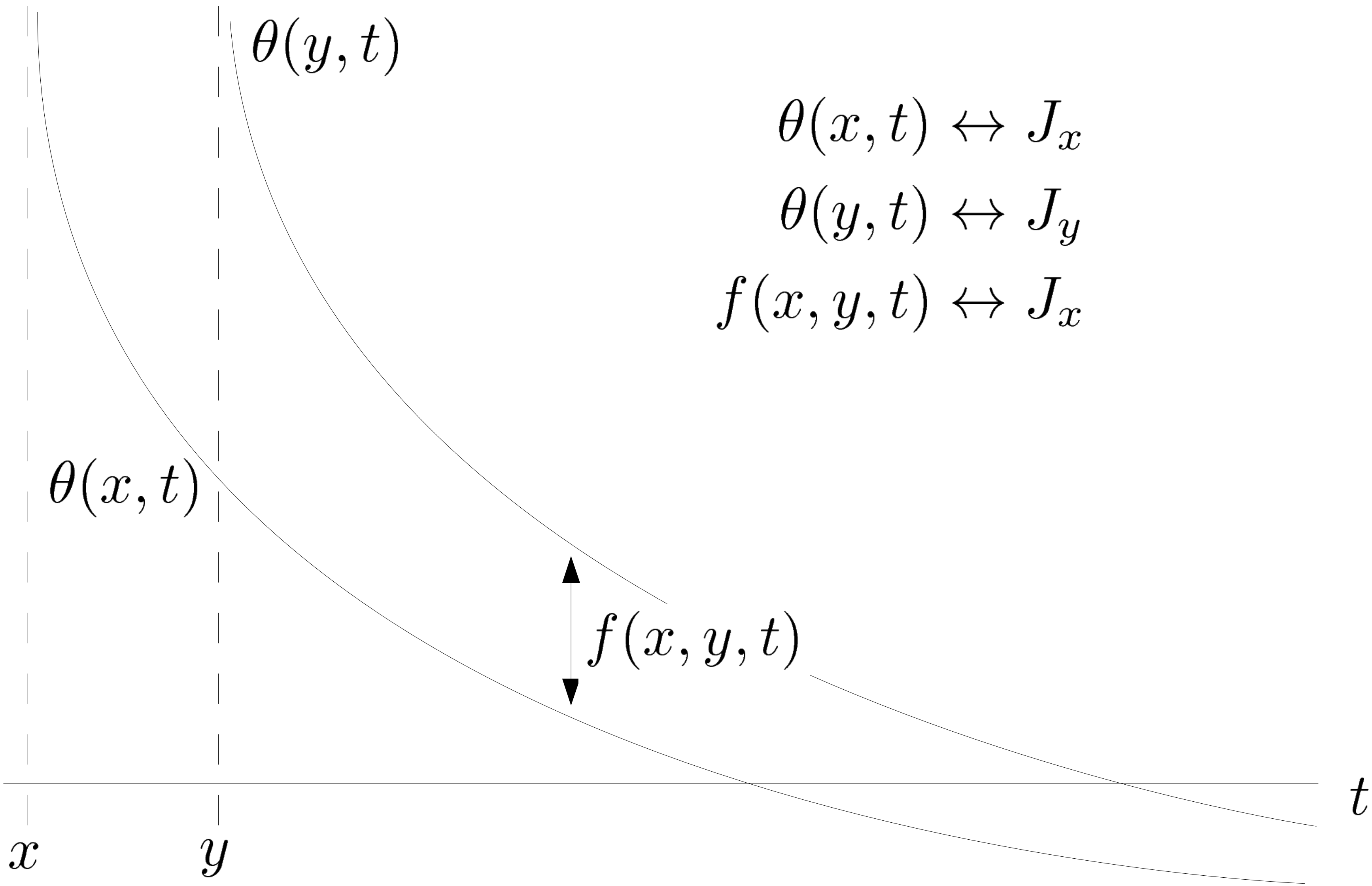
The geodesic $c : [0, a) \to M$ satisfies the Krolak condition if

for all $x \in [0, a)$ there exists $t_x \in [0, a)$

so that $\theta(x, t_x) = 0.$



An important equation



$$\theta(x,t) \leftrightarrow J_x$$
$$\theta(y,t) \leftrightarrow J_y$$
$$f(x,y,t) \leftrightarrow J_x$$

Beem, J.K., P.E. Ehrlich, and K.L. Easley, *Global Lorentzian Geometry*. Vol. 202. 1996: Marcel Dekker, Inc.

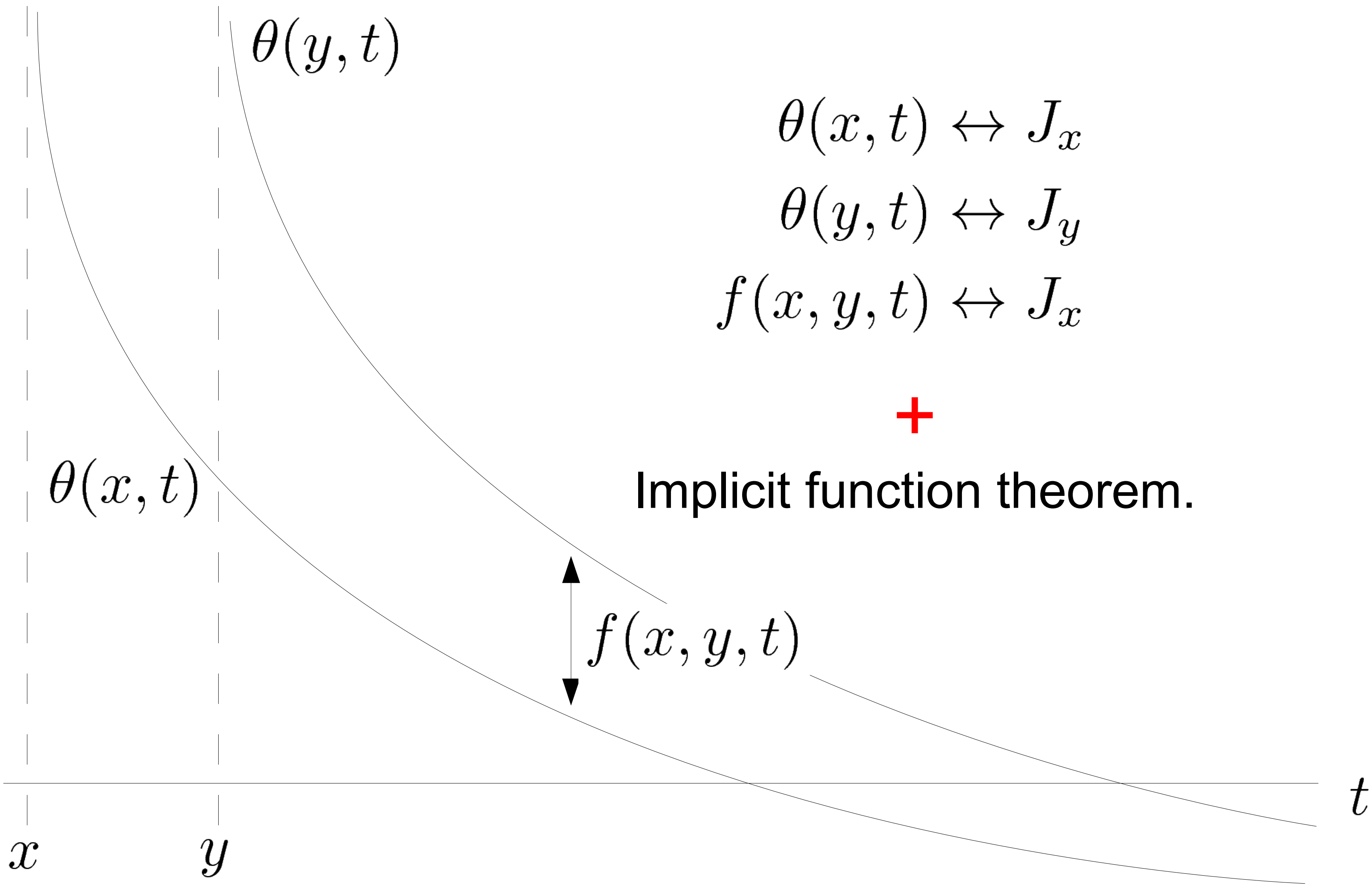
An important equation

Given J_0 we know that, for any $y \in (0, a)$,

$$\underbrace{\theta(y,t)}_{\text{function of } J_y} = \underbrace{\theta(0,t) + f(0,y,t)}_{\text{function of } J_0}.$$

Beem, J.K., P.E. Ehrlich, and K.L. Easley, *Global Lorentzian Geometry*. Vol. 202. 1996: Marcel Dekker, Inc.

New condition 1



$$\theta(x,t) \leftrightarrow J_x$$
$$\theta(y,t) \leftrightarrow J_y$$
$$f(x,y,t) \leftrightarrow J_x$$

+

Implicit function theorem.

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New condition 1

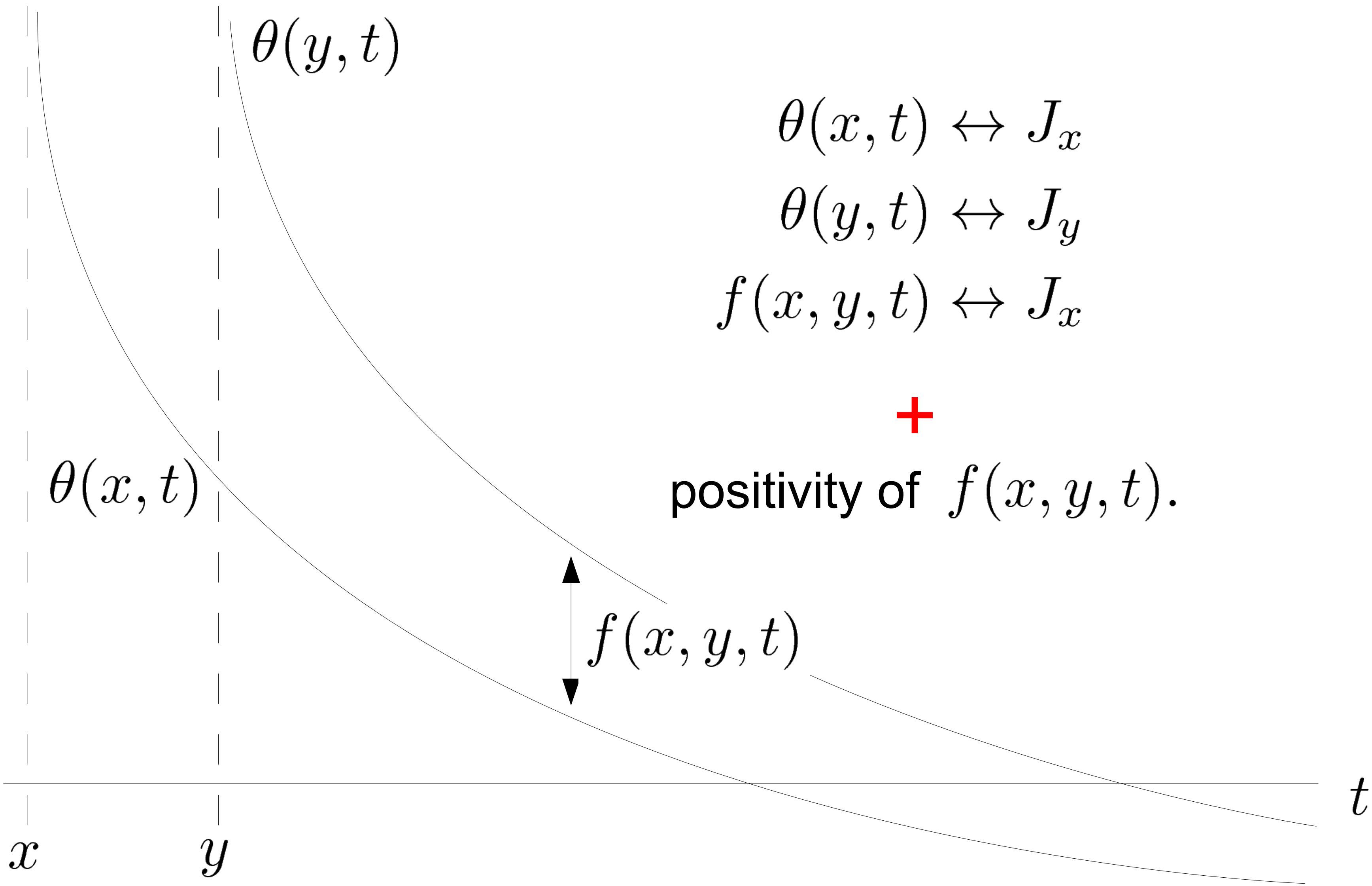
The timelike geodesic $c : [0, a) \to M$ satisfies the Krolak condition if and only if there exists a unique continuously differentiable function $g : [0, a) \to [0, a)$ that satisfies

$$G(b, g(b)) \left. \frac{d}{dt} g \right|_b + F(b, g(b)) = 0$$

for all $b \in (0, a)$ and $\theta(0, g(0)) = 0.$

$$G(y, t) \leftrightarrow \theta(0, t), \;\; f(0, y, t) \leftrightarrow J_0$$
$$F(y, t) \leftrightarrow f(0, y, t) \leftrightarrow J_0$$

New condition 2



$$\theta(x,t) \leftrightarrow J_x$$
$$\theta(y,t) \leftrightarrow J_y$$
$$f(x,y,t) \leftrightarrow J_x$$

+

positivity of $f(x,y,t)$.

Beem, J.K., P.E. Ehrlich, and K.L. Easley, *Global Lorentzian Geometry*. Vol. 202. 1996: Marcel Dekker, Inc.

New condition 2

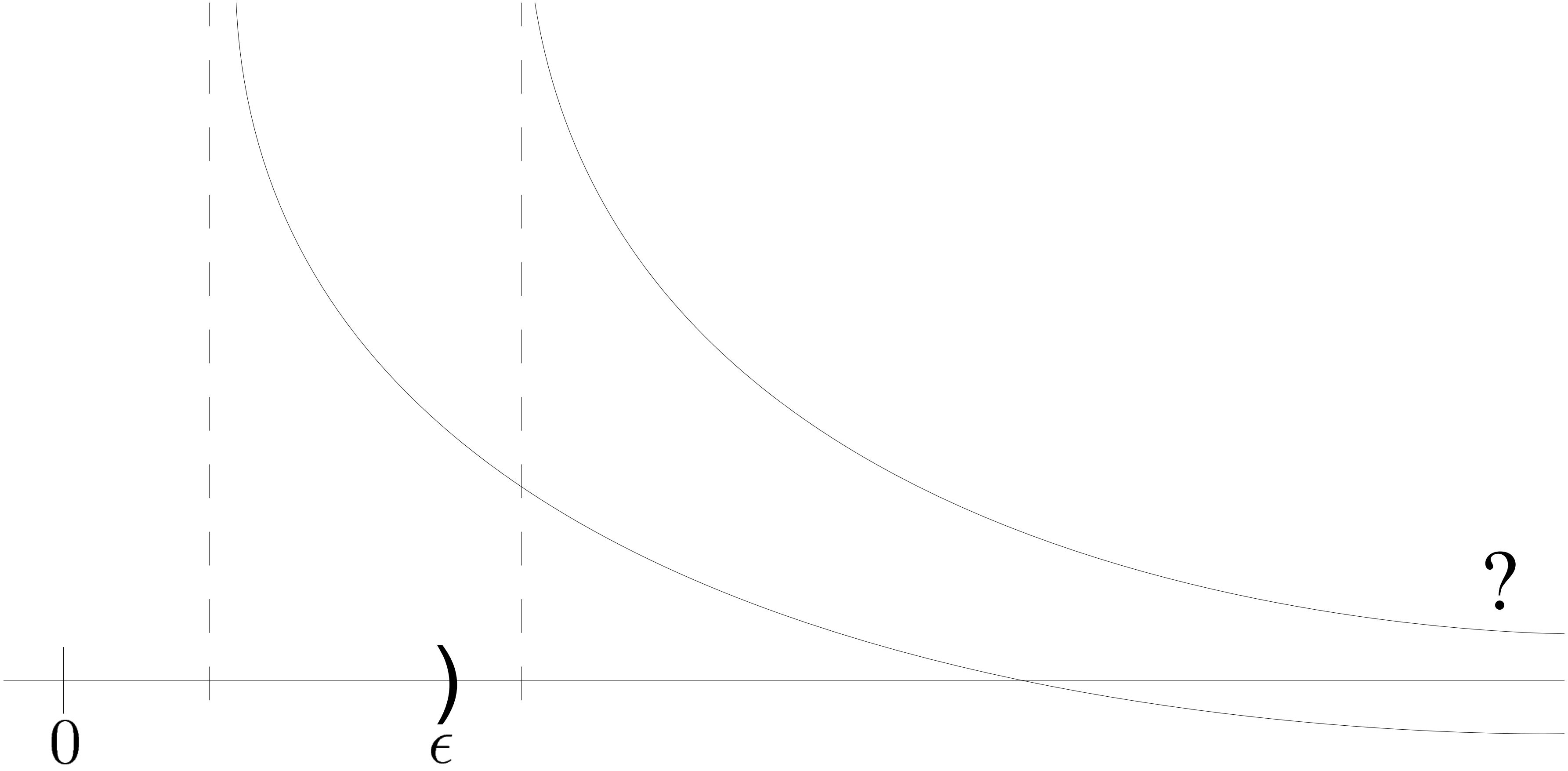
The timelike geodesic $c : [0, a) \to M$ satisfies the Krolak condition if and only if for all $b \in (0, a)$ there exists $t_b \in (b, a)$ so that,

$$-\theta(0, t_b) > f(0, b, t_b)$$

and $\theta(0, t_0) = 0$ for some $t_0 \in (0, a)$.

Links to singularity theorems

For singularity theorems with point collapse conditions we can see that,



What else is needed?

- Can the conditions be given for null geodesics?
- Can the conditions be given for surface collapse?

$$J_0(0) = I$$

$$\nabla_t J_0\big|_0 (v) = \nabla_v \eta$$

- How does $\left(J_0^T J_0\right)(\tau)$ behave in the limit?
- Lack of metric extension equivalent to certain behaviour?

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Thank you

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The difference of divergences

Let $x, y, t \in [0, a)$ with $x < y \leq t$ then $f(x, y, t)$ may be defined as

$$f(x,y,t) = \mathrm{tr}\left(\left(J_x^T J_x\right)^{-1} \left(\int_y^t \left(J_x^T J_x\right)^{-1} (\tau) d\tau \right)^{-1} \right)$$

where J_x is the Jacobi tensor with initial conditions

$$J_x(x) = 0$$

$$\nabla_t J_x \big|_x = I.$$

The all important result

Given J_0 we know that, for any $b \in (0, a)$,

$$J_b(t) = \left(J_0^T\right)(b) J_0(t) \int_b^t \left(J_0^T J_0\right)^{-1} (\tau) d\tau.$$

This gives us that,

$$\theta(b, t) = \theta(0, t) + f(0, b, t).$$

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Links to singularity theorems

For singularity theorems with point collapse conditions we can see that,

· there exists $\epsilon > 0$ so that $g : [0, \epsilon) \longrightarrow [0, a)$ satisfies the differential equation,

· there exists $\epsilon > 0$ so that for all $b \in (0, \epsilon)$ there exists $t_b \in (b, a)$ so that $-\theta(0, t_b) > f(0, b, t_b)$ and $\theta(0, t_0) = 0$ for some $t_0 \in (0, a)$.