How Did the Black Hole Get Its States?

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Black holes are thermodynamic objects

$$T = \frac{\hbar\kappa}{2\pi c}$$

$$S_{BH} = \frac{A}{4\hbar G}$$

Quantum ($\hbar$) and gravitational (G)

Does this thermodynamic behavior have a microscopic explanation?

Outline

1. Black hole entropy and the problem of universality
2. Symmetries and state-counting:
   conformal field theory and the Cardy formula
3. The BTZ black hole
4. Horizon constraints
5. Dilaton black holes and the Bekenstein-Hawking entropy

The problem of “universality” of black hole entropy

Black hole entropy counts:

- Weakly coupled string and D-brane states
- States in a dual conformal field theory “at infinity”
- Spin network states crossing the horizon
- “Heavy” degrees of freedom in induced gravity
- No local states—it’s inherently global
- Nothing—it comes from quantum field theory in a fixed background, and doesn’t know about quantum gravity

Answer: apparently, all of the above

Is there an underlying symmetry that can explain why these approaches all agree?

A small detour: entropy and the Cardy formula

Any two-dimensional conformal field theory
can be characterized by generators $L[\xi]$ and $\bar{L}[\bar{\xi}]$
of holomorphic and antiholomorphic diffeomorphisms

Virasoro algebra:

$$[L[\xi], L[\eta]] = L[\eta\xi' - \xi\eta'] + \frac{c}{48\pi}\int dz\,(\eta'\xi'' - \xi'\eta'')$$

Central charge c (“conformal anomaly”) depends on theory

Consider a conformal field theory with

- central charge c
- lowest eigenvalue Δ_0 of $L[\xi_0]$

Then for large L_0, the density of states is asymptotically

$$\boxed{\ln \rho(L_0) \sim 2\pi\sqrt{\frac{(c - 24\Delta_0)L_0}{6}}}$$

Entropy is fixed by symmetry, independent of details!

Why this might help:
matter near a horizon looks conformal

Black hole in “tortoise” coordinates:

$$ds^2 = N^2(dt^2 - dr_*{}^2) + ds_\perp{}^2$$

($N \longrightarrow 0$ at horizon)

Scalar field:

$$(\Box - m^2)\varphi = \frac{1}{N^2}(\partial_t^2 - \partial_{r_*}^2)\varphi + O(1)$$

Mass and transverse excitations become negligible
Effective two-dimensional conformal field

Can this symmetry be extended to the gravitational field?
Medved, Martin, Visser: conformal symmetry at Killing horizon

The BTZ black hole

Bañados, Teitelboim, and Zanelli:

Black hole solution of (2+1)-d vacuum Einstein equations
with cosmological constant $\Lambda = -1/\ell^2$

"Three-dimensional Kerr black hole"

$$ds^2 = -(N^\perp)^2 dt^2 + f^{-2} dr^2 + r^2 \left(d\phi + N^\phi dt\right)^2$$

with $N^\perp = f = \left(-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}\right)^{1/2}, \quad N^\phi = -\frac{J}{2r^2}$

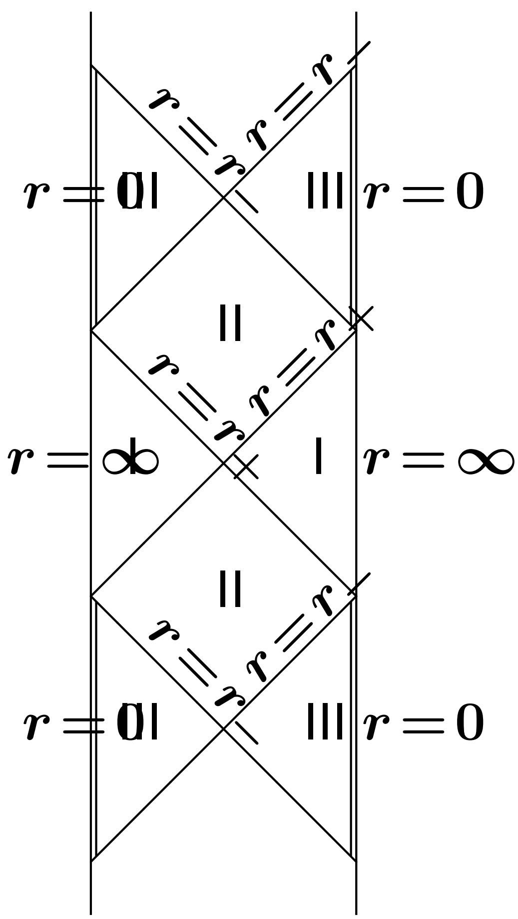
Event horizon at $r = r_+$
Inner horizon at $r = r_-$

$$M = \frac{r_+^2 + r_-^2}{8G\ell^2}$$

$$J = \frac{r_+ r_-}{4G\ell}$$

The BTZ black hole is

- a constant curvature spacetime
- a quotient of anti-de Sitter space by a group of isometries
- a genuine black hole:
  - true event horizon
  - inner Cauchy horizon
  - can form from collapsing matter
  - standard black hole thermodynamics



Carter-Penrose diagram for BTZ black hole

A Mystery:

– The BTZ black hole has standard Bekenstein-Hawking entropy, $S = 2\pi r/4\hbar G$

– *but* (2+1)-dimensional gravity has no dynamical degrees of freedom!

Where did the entropy come from?

Partial Answer
(Strominger; Birmingham, Sachs, and Sen):

Generators of diffeomorphisms that preserve boundary conditions satisfy a Virasoro algebra with classical central charge $c = 3\ell/2G$ (Brown and Henneaux)

For the BTZ black hole

$$L_0 \sim \frac{1}{16G\ell}(r_+ + r_-)^2, \quad \bar{L}_0 \sim \frac{1}{16G\ell}(r_+ + r_-)^2$$

By Cardy formula,

$$S \sim \frac{2\pi}{8G}(r_+ + r_-) + \frac{2\pi}{8G}(r_+ - r_-) = \frac{2\pi r_+}{4G}$$

Correct Bekenstein-Hawking entropy

Entropy is related to “boundary conditions” at infinity…

Another Piece of the Answer

(Witten; Elitzur, Moore, and Seiberg; S.C.)

- (2+1)-dimensional gravity with $\Lambda < 0$ can be written as an $\mathbf{SO(2,1)} \times \mathbf{SO(2,1)}$ Chern-Simons theory
- C-S action is gauge-invariant *on a compact manifold*
- Boundaries and required boundary conditions can partially break that invariance

Boundary conditions $\Rightarrow$ broken gauge invariance $\Rightarrow$ "pure gauge" states become physical

More details:

Effective boundary theory is a Liouville theory
(Coussaert, Henneaux, van Driel; Bautier et al.)

Chen: Liouville states can be counted, give correct entropy

Why this isn't good enough

BTZ black hole analysis gives nice picture of microstates
But. . .

- States should be at horizon, not infinity
- Chern-Simons picture works only in 2+1 dimensions
- Boundary is two-dimensional only in 2+1 dimensions

Possible lessons:

- Look for "broken gauge invariance"
- Hope for an effective two-dimensional picture
- But look near horizon

Horizons and constraints

or

How do you ask about a black hole in quantum gravity?

Standard approach:
Fix black hole background, ask about quantum fields,
gravitational perturbations, etc.

You can’t do that in quantum gravity!

Alternative:
Ask a conditional question. . .
impose black hole characteristics as constraints

For example:
– Restrict path integral to metrics with horizons, or
– Add constraints to canonical theory requiring a horizon

Two-dimensional dilaton gravity with horizon constraints

Null frame $\{l^a, n^a\}$, $l \cdot n = -1$

with $l = \sigma du + \alpha dv, \quad n = \beta du + \tau dv$

Dilaton A, "surface gravities"

$$\nabla_a l_b = -\kappa n_a l_b - \bar{\kappa} l_a l_b$$
$$\nabla_a n_b = \kappa n_a n_b + \bar{\kappa} l_a n_b$$

Action $I = \dfrac{1}{16\pi G} \displaystyle\int d^2x \sqrt{-g}\,[AR + V(A)]$

Treat u as time coordinate, v as space coordinate;
do standard canonical decomposition. . .

Find constraints

$$C_\perp = \frac{1}{2}\pi_\alpha \pi_A - {\pi_\alpha}' + \tau V(A)$$
$$C_\| = \alpha {\pi_\alpha}' + \tau {\pi_\tau}' - \pi_A A'$$
$$C_\pi = \tau \pi_\tau - \alpha \pi_\alpha + 2A'$$

Horizon constraints:

$u = 0$ is a horizon ("nonexpanding horizon") with null normal l if

- $\alpha = 0$ (so l really is the normal);
- expansion $\vartheta = l^a\nabla_a A/A = 0$

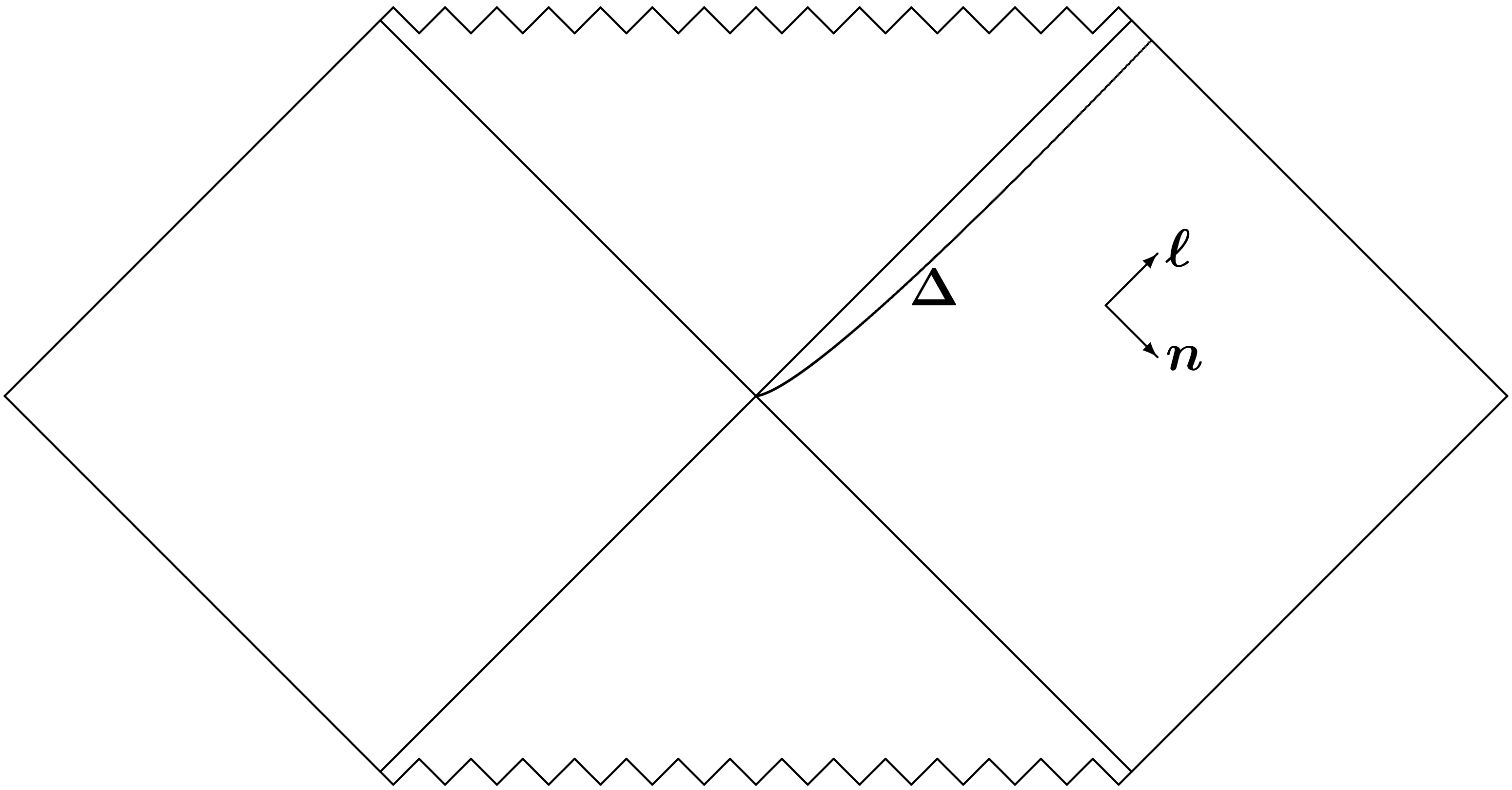
But... this requires light cone quantization (initial surface is null); too hard for now

"Stretched horizon":

- $\alpha = \epsilon_1 \ll 1$ ("almost null");
- $l^v\nabla_v A/\kappa A = \epsilon_2 \ll 1$ ("almost nonexpanding")

Stretched horizon constraints:

$$K_1 = \alpha - \epsilon_1 = 0$$
$$K_2 = A' - \frac{1}{2}\epsilon_2 A_+\pi_A + \frac{a}{2}C_\pi = 0$$



Dilaton black hole with “stretched horizon” $\boldsymbol{\Delta}$

The K_i are second class constraints ($\{K_i, K_j\} \not\approx 0$)

Need to make sure they are preserved by symmetries:

$$C \to C^* = C + a_1 K_1 + a_2 K_2$$

with $\{C^*, K_i\} \approx 0$

Dirac brackets:

$$\begin{aligned}
\{C_{||}[\xi], C_{||}[\eta]\}^* &= -C_{||}[\xi\eta' - \eta\xi'] \\
&\quad - \frac{2(1+a)}{a^2}\epsilon_2 A_+ \int dv(\xi'\eta'' - \eta'\xi'') \\
\{C_{||}[\xi], C_\pi[\eta]\}^* &= -C_\pi[\xi\eta'] \\
&\quad - \frac{2}{a}\left(\frac{2}{a} + 1\right)\epsilon_2 A_+ \int dv\, \xi'\eta' \\
\{C_\pi[\xi], C_\pi[\eta]\}^* &= -\frac{2}{a^2}\epsilon_2 A_+ \int dv(\xi\eta' - \eta\xi')
\end{aligned}$$

Choose $a = -2$:

then $C_{||}$ generates a Virasoro algebra with central charge

$$c = 24\pi\epsilon_2 A_+$$

and C_π is a primary field of weight one.

Boundary term:

For $\delta C_{\|}^{*}$ to be well-defined, need boundary term

$$\boldsymbol{C_{\partial} = (\xi \pi_A A + \xi' A - \xi A')\big|_{v=v_+}}$$

Modes:

$$\boldsymbol{z = e^{2\pi i A/A_+}, \qquad \xi_n = \frac{A_+}{2\pi A'} z^n}$$

Virasoro generator:

$$\boldsymbol{\Delta = C_{\|}^{*}[\xi_0] \approx \frac{A_+}{\pi \epsilon_2} + \mathcal{O}(1)}$$

Cardy formula:

$$\boldsymbol{S = \ln \rho(\Delta) = 2\pi \sqrt{\frac{c L_0}{6}} = \frac{A_+}{4\hbar G} + \mathcal{O}(\epsilon_2)}$$

Correct Bekenstein-Hawking entropy

What are the relevant states?

Standard approach to canonical gravity: require that

$$C_{||}|phys\rangle = C_{\pi}|phys\rangle = 0$$

This is not consistent with Virasoro algebra with $c \neq 0$

$$[C^*, C^*]|phys\rangle \sim C^*|phys\rangle + const.|phys\rangle$$

Need to weaken constraints:

$$C_{||}^{*(+)}|phys\rangle = 0 \Leftrightarrow \langle phys|C_{||}^{*}|phys\rangle = 0$$

States that were formerly “pure gauge”
now become physical. . .

Open Questions

1. Can this be extended beyond two dimensions?
2. How sensitive are results to details of “stretched horizon”? Light cone quantization might give an answer.
3. How do these results relate to past work on near-horizon symmetries?
4. Can these results be related to path integral methods? Euclidean black hole analysis might help. . .
   ($\boldsymbol{K_2}$ constraint $\Leftrightarrow$ circle of constant proper radius)