Global and local problems with Kerr's solution.

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Contents

1. Conclusions of Roy Kerr's PRL 11, 237-8 (1963).

" Among the solutions ... there is one which is stationary ... and also axisymmetric. Like the Schwarzschild metric, which it contains, it is type D ... m is a real constant ... The metric is

$$\mathrm{d}s^2 = (r^2+a^2\cos^2\theta)(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)+2(\mathrm{d}u+a\sin^2\theta\,\mathrm{d}\phi)$$

$$\times(\mathrm{d}r+a\sin^2\theta\,\mathrm{d}\phi)-\left(1-\frac{2mr}{r^2+a^2\cos^2\theta}\right)(\mathrm{d}u+a\sin^2\theta\,\mathrm{d}\phi)^2\,,$$

where a is a real constant. This may be transformed to an asymptotically flat coordinate system ... we find that m is the Schwarzschild mass and ma the angular momentum ... It would be desirable to calculate an interior solution ... "

2. Transformation to Kerr - Schild form.

In his 1963 PRL and with *Schild* in sequel, *Kerr* obtained form

$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu$ with $g_{\mu\nu} = \eta_{\mu\nu} + 2(m/U)n_\mu n_\nu$ in **flat** background with **null** covector $n_\mu \mathrm{d}x^\mu = \mathrm{d}u + a\sin^2\theta\,\mathrm{d}\phi$, for $U = (r^2 + a^2\cos^2\theta)/r$, by setting $\bar{t} = u - r$, $\bar{z} = a\cos\theta$, $\bar{x} + i\bar{y} = (r - ia)\mathrm{e}^{i\phi}\sin\theta$, which gives

$$n_\mu \mathrm{d}x^\mu = \mathrm{d}\bar{t} + \frac{\bar{z}\mathrm{d}\bar{z}}{r} + \frac{(r\bar{x} - a\bar{y})\mathrm{d}\bar{x} + (r\bar{y} + a\bar{x})\mathrm{d}\bar{y}}{r^2 + a^2},$$

in Minkowski coords, $\eta_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}\bar{x}^2 + \mathrm{d}\bar{y}^2 + \mathrm{d}\bar{z}^2 - \mathrm{d}\bar{t}^2$.

Generalised to higher dim. by *Myers & Perry* ('86), also to **De Sitter** background, in 4 dim. by self (*Les Houches*, '72), 5 dim. by *Hawking, Hunter, Taylor Robinson* (hep-th/9811056) and higher dim. by *Gibbons, Pope, Page* (hep-th/0404008).

3. Transformation to Boyer - Lindquist form.

As well as time and axial sym, Kerr solution has discrete PT sym predicted by "circularity" theorem (*Papapetrou* '66) and made manifest in '67 by *Boyer & Lindquist* transformation

$$\mathrm{d}t = \mathrm{d}u - (r^2 + a^2)\Delta^{-1}\mathrm{d}r\,, \quad \mathrm{d}\varphi = -\mathrm{d}\phi + a\,\Delta^{-1}\mathrm{d}r\,,$$

with $\Delta = r^2 - 2mr + a^2$, giving Kerr's null form as

$$n_\mu \mathrm{d}x^\mu = \mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi + (r^2 + a^2\cos^2\theta)\Delta^{-1}\mathrm{d}r\,,$$

and metric $$\mathrm{d}s^2 = (r^2 + a^2\cos^2\theta)(\Delta^{-1}\mathrm{d}r^2 + \mathrm{d}\theta^2)$$

$$+(r^2 + a^2)\sin^2\theta\,\mathrm{d}\varphi^2 + \frac{2mr(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi)^2}{r^2 + a^2\cos^2\theta} - \mathrm{d}t^2\,,$$

having no cross terms with non-ignorable diffs, $\mathrm{d}r$ or $\mathrm{d}\theta$, but coord singy if $a^2 \leq m^2$ on null "horizon" where Δ vanishes.

4. Transformation to canonical separable form.

As well as ordinary "circular" sym generated by Killing vectors, $k^\mu\partial/\partial x^\mu = \partial/\partial t$ and $h^\mu\partial/\partial x^\mu = \partial/\partial\varphi$, Kerr metric has hidden symmetry revealed (*B.C.* '68) by **canonical** tetrad $g_{\mu\nu} = \sum_{i=1}^{3} \vartheta^{\hat{i}}_{\mu}\vartheta^{\hat{i}}_{\nu} - \vartheta^{\hat{0}}_{\mu}\vartheta^{\hat{0}}_{\nu}$ specified with $\Delta = r^2 - 2mr + a^2$, $\varrho = \sqrt{r^2 + a^2\cos^2\theta}$ by $\vartheta^{\hat{1}}_{\mu}\mathrm{d}x^\mu = (\varrho/\sqrt{\Delta})\mathrm{d}r$, $\vartheta^{\hat{2}}_{\mu}\mathrm{d}x^\mu = \varrho\,\mathrm{d}\theta$,

$$\frac{\vartheta^{\hat{3}}_{\mu}\mathrm{d}x^\mu}{\sin\theta} = \frac{(r^2+a^2)\mathrm{d}\varphi - a\mathrm{d}t}{\varrho}, \quad \frac{\vartheta^{\hat{0}}_{\mu}\mathrm{d}x^\mu}{\sqrt{\Delta}} = \frac{\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi}{\varrho},$$

giving Kerr-Schild form $g_{\mu\nu} = \eta_{\mu\nu} + 2mr(\vartheta^{\hat{0}}_{\mu} + \vartheta^{\hat{1}}_{\mu})(\vartheta^{\hat{0}}_{\nu} + \vartheta^{\hat{1}}_{\nu})$ and with Killing-Yano 2-form (*Penrose* '73) given by

$$f_{\mu\nu} = 2a\cos\theta\,\vartheta^{\hat{1}}_{[\mu}\vartheta^{\hat{0}}_{\nu]} - 2r\,\vartheta^{\hat{2}}_{[\mu}\vartheta^{\hat{3}}_{\nu]}\,.$$

5. Killing-Yano symmetry for scalar fields.

Killing-Yano condition $\nabla_\mu f_{\nu\rho} = \nabla_{[\mu} f_{\nu\rho]}$ provides solution $K_{\mu\nu} = f_{\mu\rho} f^\rho{}_\nu$ of Killing tensor eqn $\nabla_{(\mu} K_{\nu\rho)} = 0$ as well as secondary and primary solutions $\tilde{k}^\mu = K^\mu{}_\nu k^\nu = a^2 k^\mu + a h^\mu$ and $k^\mu = \frac{1}{6}\varepsilon^{\mu\nu\rho\sigma}\nabla_\nu f_{\rho\sigma}$ of Killing vector eqn $\nabla_{(\mu} k_{\nu)} = 0$.

For affine geodesic motion, $p^\nu \nabla_\nu p^\mu = 0$, these give (energy and axial angular momentum) consts $\mathcal{E} = k^\nu p_\nu$ and $\mathcal{M} = h^\nu p_\nu$, while Killing tensor gives const $\mathcal{K} = K^{\mu\nu} p_\mu p_\nu = \ell_\mu \ell^\nu$, with (angular momentum) $\mathcal{J}_\mu = f_{\mu\nu} p^\mu$ obeying $p^\nu \nabla_\nu \ell_\mu = 0$.

Get corresp. (self adjoint) operators $\mathcal{E} = i k^\nu \nabla_\nu$, $\mathcal{M} = i h^\nu \nabla_\nu$, and $\mathcal{K} = \nabla_\mu K^{\mu\nu} \nabla_\nu$ that commute with Dalembertian $\Box = \nabla^\nu \nabla_\nu$ on scalar field : $[\mathcal{E}, \Box] = 0$, $[\mathcal{M}, \Box] = 0$, and (with integrability cond. $K^\rho{}_{[\mu} R_{\nu]\rho} = 0$) also $[\mathcal{K}, \Box] = 0$.

6. Killing-Yano symmetry for spinor fields.

Ensuing integrability of geodesic and scalar wave equations equiv to their solubility by separation of variables (*B.C.* '68). Extension to massless (*Unruh* '73) and massive (*Chandra* '76) spin 1/2 field governed by Dirac operator $\mathcal{D} = \gamma^\mu \nabla_\mu$ due to corresp. spinor operator conservation laws of energy, $[\mathcal{E}, \mathcal{D}] = 0$, axial ang. mom. $[\mathcal{M}, \mathcal{D}] = 0$, and (unsquared) total ang mom $[\mathcal{J}, \mathcal{D}] = 0$ (*B.C., R. McL '79*) with $\mathcal{E} = i k^\nu \nabla_\nu + \frac{1}{4} i (\nabla_\mu k_\nu) \gamma^\mu \gamma^\nu$, $\mathcal{M} = i h^\nu \nabla_\nu + \frac{1}{4} i (\nabla_\mu h_\nu) \gamma^\mu \gamma^\nu$, and $\mathcal{J} = i \gamma^\mu (\gamma^5 f_\mu{}^\nu \nabla_\nu - k_\mu)$.

Such a neat commutation formulation is not (yet ?) available for extension (*Teukolsky* '73) of solubility by separation of variables to massless spin 1 (electromag) and – particularly important for demo (*Whiting* '89) of stability – to spin 2 (grav) perturbations.

7. The black hole property for $a^2 \le m^2$.

The feature first clearly recognised by *Boyer* '65 making Kerr metric so important is **black hole** property for $a^2 \le m^2$: "asymptopia" both visible and accessible only in non sing region bounded by past and future null (outer) **horizons** on which $\Delta = 0$ where $r = m + c$, $c = \sqrt{m^2 - a^2}$. Topology within first elucidated on axis using conformal proj (*B.C.* '66) and then completely by *Boyer & Lindquist* '67, *B.C.* '68.

There is always a causally pathological "time machine" region, including irremovable **ring singy** where $r^2 + a^2 \cos^2\theta \longrightarrow 0$, but it extends to asympt flat region only in (presumably unphysical) case for which $a^2 > m^2$ – being otherwise confined by (unstable) "inner horizon", where $r = m - c$.

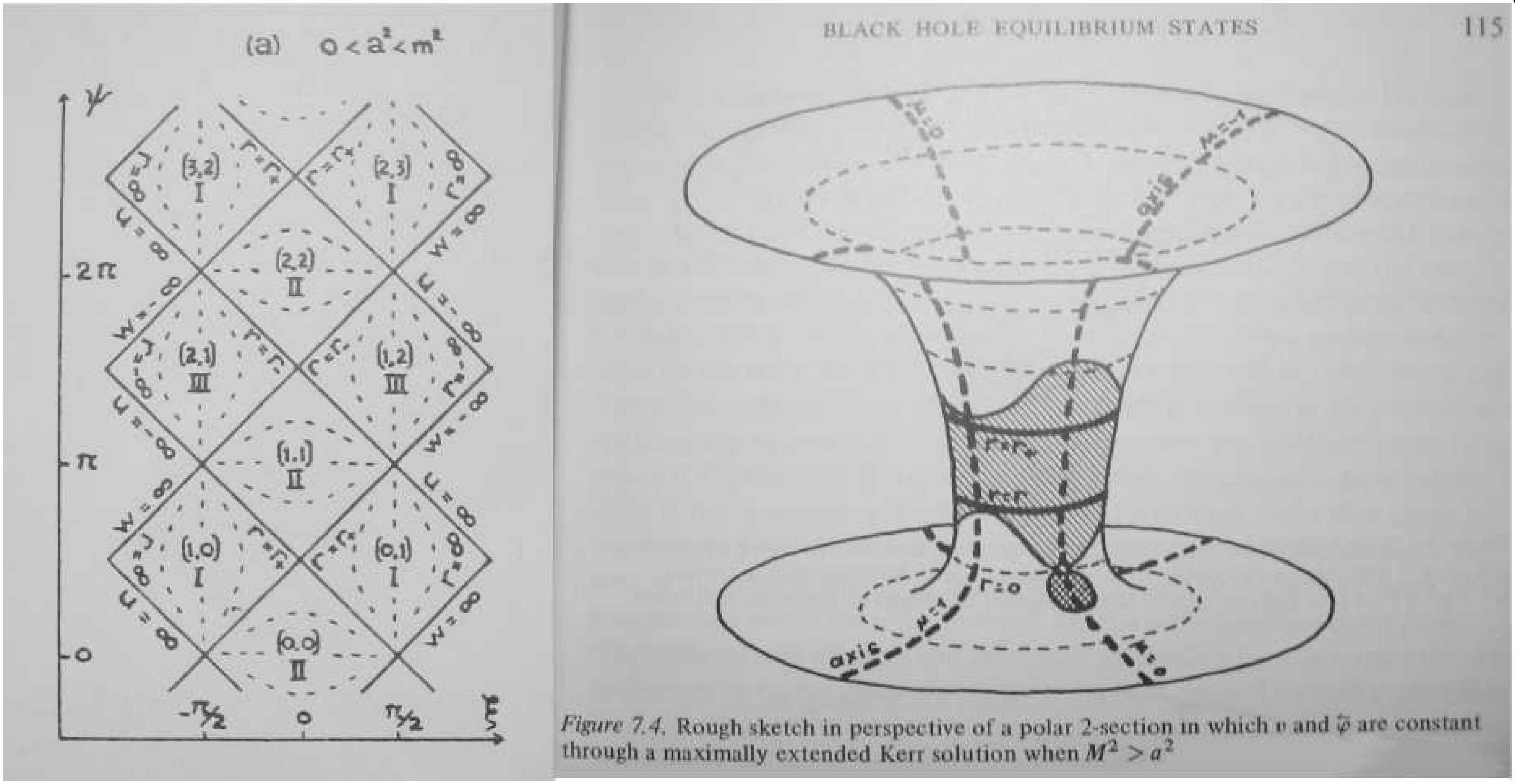

FIG. 1 – (a) Conformal projection of symmetry axis (*B.C.* '66) ; (b) polar section from pos to neg r region with shading of "ergoregion" and "time machine" (*Les Houches* '72).

8. The black hole equilibrium problem

Overwhelming importance of Kerr soln derives from its provision of generic representation of final outcome of grav collapse, as implied by **no-hair theorem** (*B.C.* '71) – proving no other vacuum black hole equilibrium state obtainable by continuous axisym variation from spherical Schwarzschild soln (shown by previous work of *Israel* '67 to be only static possibility).

Confirmation by subsequent work (*Hawking* '72, *Robinson '74 Crusciel & Wald* '95) towards unrestricted black hole **uniqueness theorem** (though some loose ends remain concerning assumptions of analyticity and causality) and work (by *Vishveshwara* '70 and *Whiting* '89) refuting earlier conjecture (*Israel* '67) of dynamical instability.

9. Demonstration of no-hair (& uniqueness) theorem

Uses (ellipsoidal) $\mathrm{d}\hat{s}^2 = \mathrm{d}\lambda^2/(\lambda^2 - c^2) + \mathrm{d}\mu^2/(1-\mu^2)$, generic stationary axisym asympt flat vacuum metric given by $\mathrm{d}s^2 = \varrho^2 \mathrm{d}\hat{s}^2 + X(\mathrm{d}\varphi - \omega \mathrm{d}t)^2 - (\lambda^2 - c^2)(1-\mu^2)\mathrm{d}t^2$ (*Papapetrou* '66) with Einstein eqns from (pos def) action $\int \mathrm{d}\lambda\, \mathrm{d}\mu (|\hat{\nabla} X|^2 + |\hat{\nabla} Y|^2)/X^2$, using potential (*Ernst* '68) given by $X^2 \partial\omega/\partial\lambda = (1-\mu^2)\partial Y/\partial\mu$. Need regularity on horizon (with rigid ang vel Ω) where $\lambda = c$ and appropriate boundary conditions (*B.C.* '71) on axis where $\mu = \pm 1$ and at large radius $\lambda \to \infty$ in terms of angular momentum ma .

Non trivial divergence identities (*B.C.* '71, *Robinson* '74, *Mazur* '82, *Bunting* '83) lead (with $\lambda = r - m$, $\mu = \cos\theta$) to Kerr solution having mass $m = \sqrt{c^2 + a^2}$ and horizon angular velocity $\Omega = a/2m(m+c)$.