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Kerr black hole and rotating wormhole

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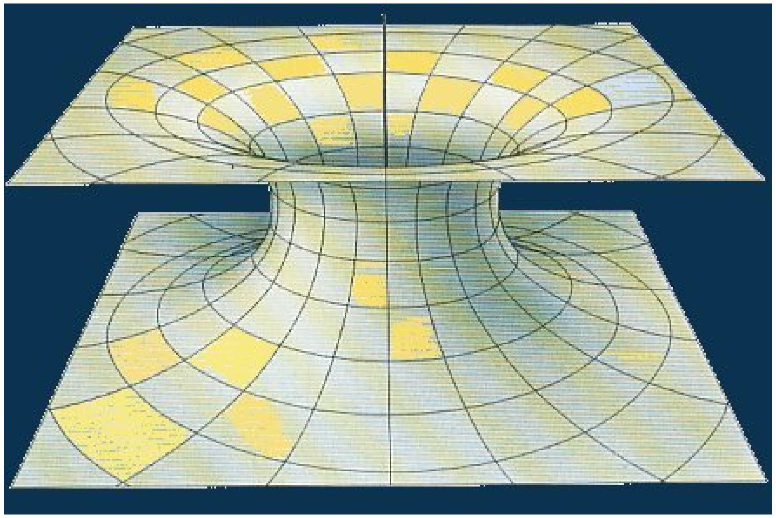
Introduction

The wormhole structure: two asymptotically flat regions + a bridge



- To be traversable: exotic matter which violates the known energy conditions
- Exotic matter is also an important issue on dark energy which accelerates our universe.
- Requirement of the more general wormhole model - 'rotating wormhole'
- Two-dimensional model of transition between black hole and wormhole $\Rightarrow$ Interest on the general relation between black hole and wormhole

Static Wormhole (Morris and Thorne, 1988)

The spacetime metric for static wormhole

$$ds^2 = -e^{2\Lambda(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$\Lambda(r)$: **the lapse function**

$b(r)$: **wormhole shape function**

At t =**const. and** $\theta = \pi/2$**, the 2-curved surface is embedded into 3-dimensional Euclidean space**

$$d\tilde{s}^2 = \frac{dr^2}{1 - b(r)/r} + r^2 d\phi^2 = dz^2 + dr^2 + r^2 d\phi^2$$

Flare-out condition

$$\frac{d^2r}{dz^2} = \frac{b - b'r}{2b^2} > 0$$

With new radial coordinate $l \in (-\infty, \infty)$ **(proper distance), while** $r > b$

$$ds^2 = -e^{2\Lambda(l)}dt^2 + dl^2 + r(l)^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where

$$\frac{dl}{dr} = \pm\left(1 - \frac{b}{r}\right)^{-1/2}$$

Rotating Wormhole (Teo, 1998)

The spacetime in which we are interested will be stationary and axially symmetric.

The most general stationary and axisymmetric metric can be written as

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{ij}dx^i dx^j,$$

where the indices $i, j = 1, 2$.

Freedom to cast the metric into spherical polar coordinates by setting $g_{22} = g_{\phi\phi}/\sin^2 x^2 \quad \rightarrow$

The metric of the rotating wormhole as:

$$\begin{aligned} ds^2 &= -N^2dt^2 + e^{\mu}dr^2 + r^2K^2d\theta^2 + r^2K^2\sin^2\theta[d\phi^2 - \Omega dt]^2 \\ &= -N^2dt^2 + \frac{dr^2}{1-b(r)/r} + r^2K^2d\theta^2 + r^2K^2\sin^2\theta[d\phi^2 - \Omega dt]^2, \end{aligned}$$

where Ω is the angular velocity $d\phi/dt$ acquired by a particle that falls freely from infinity to the point (r, θ), and which gives rise to the well-known dragging of inertial frames or Lense-Thirring effect in general relativity.

N, b, K, Ω**: functions of both r and θ.**

$K(r, \theta)$**: a positive, nondecreasing function that determines the "proper radial distance" R measured at (r, θ) from the origin.**

- Regular on the symmetry axis $\theta = 0, \pi$

- Two identical, asymptotically flat regions joined together at throat $r = b$.

- No event horizons or curvature singularities.

- $T_{t\phi}$: the rotation of the matter distribution.

i) In the limit of zero rotation and spherical symmetry:

$$N(r,\theta) \rightarrow e^{\Lambda(r)}, \quad b(r,\theta) \rightarrow b(r), \quad K(r,\theta) \rightarrow 1, \quad \Omega(r,\theta) \rightarrow 0.$$

$\Rightarrow$ the MT metric

ii) As $r \rightarrow \infty$.

$$N \rightarrow 1, \quad \frac{b}{r} \rightarrow 0, \quad K \rightarrow 1, \quad \Omega \rightarrow 0$$

$\Rightarrow$ Asymptotically flat

Thus, r is asymptotically the proper radial distance.

In particular, if

$$\Omega = \frac{2a}{r^3} + O\left(\frac{1}{r^4}\right),$$

then by changing to Cartesian coordinates, it can be checked that a is the total angular momentum of the wormhole.

Source of the rotating wormhole–**Bergliaffa and Hibberd(2000)**

The constraints on the Einstein tensor that arise from the matter used as a source of Einstein's equation for a generally axially symmetric and stationary spacetime.

$$G_{11} - G_{22} = 0, \quad G_{12} = 0, \quad G_{03}^2 = (G_{00} + G_{22})(G_{11} - G_{33})$$

and

$$G_{00} + G_{33} \geq 2G_{03}, \quad \left(\frac{u_0}{u_3}\right)^2 > 1,$$

where u_μ **is the four-velocity of the fluid model as the stress-energy tensor.**

Example)

1) The first example(Teo)

$$N = K = 1 + \frac{(4a\cos\theta)^2}{r}, \quad b = 1, \quad \Omega = \frac{2a}{r^3}.$$

Note that a is the angular momentum of the resulting wormhole.

Proper radius $R = 1 + (4a\cos\theta)^2$ is at throat $r = 1$

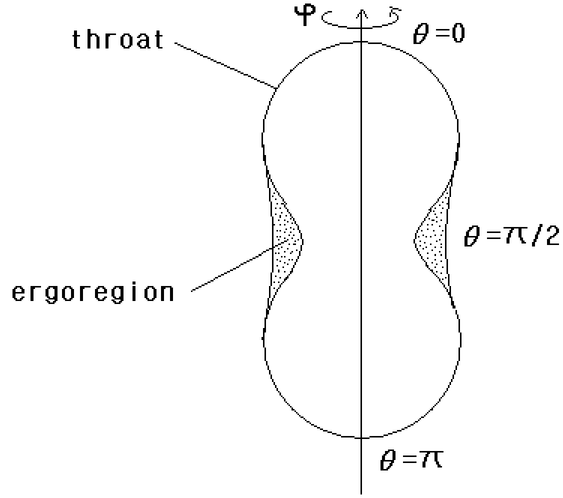
Proper distance

$$l = \pm[\sqrt{r(r-1)} + \ln(\sqrt{r} + \sqrt{r-1})].$$

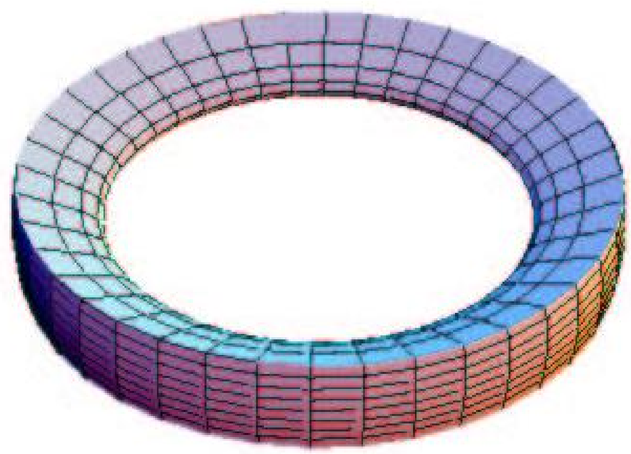
If the rotation of the wormhole is sufficiently fast, $g_{tt} = -(N^2 - \Omega^2 r^2 \sin^2\theta)$ becomes positive in some region outside the throat, indicating the presence of an ergoregions where particles can no longer remain stationary with respect to infinity.

This occurs when $r^2 = |2a\sin\theta| > 1$ and $|a| > 1/2$.

The ergoregions does not completely surround the throat, but forms a "tube" around the equatorial region instead of ergosphere in Kerr metric.



Plot of the cross-sectional schematic of the rotating wormhole throat



Ergoregion for the dumbbell-like throat wormhole model. It forms a tube.

2) The second example,

Rigid rotating wormhole, i.e.,

$$N = K = 1, \quad b = \frac{b_0^2}{r}, \quad \Omega = \text{const.}$$

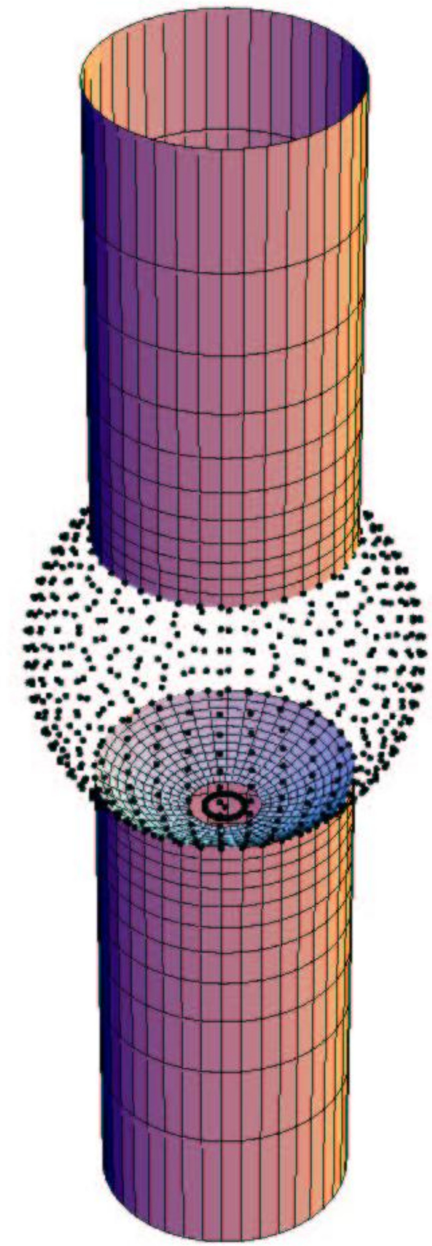
$$ds^2 = -dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta [d\phi - \Omega dt]^2$$

Without the angular velocity, the spacetime is just the simplest wormhole model used in former papers.

The throat is at $r = b_0$ and the shape of the throat is a sphere.

The ergoregion is when $r = 1/|\Omega \sin\theta|$.

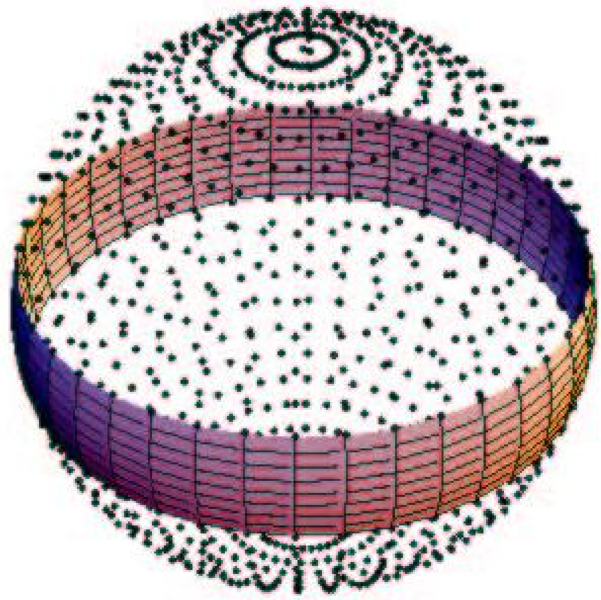
When $b_0\Omega > 1$, the ergoregion is cut by throat, while the cylinder shape ergoregion cover the whole wormhole throat, when $b_0\Omega \leq 1$.



Ergoregion of the rigid rotating wormhole model with constant angular velocity, when $b_0\Omega > 1$. The dotted region is the wormhole throat.

3) The third example,

$\Omega = 2a/r^3$ and when $r > b_0$, the ergoregion is tube type for large angular momentum. It is similar to Example 1.



Ergoregion of the rigid rotating wormhole model with $\Omega = 2a/r^3$. The dotted region is the wormhole throat. The ergoregion also forms a tube.

Kerr metric

For stationary axisymmetric spacetime metric

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\psi}(d\phi - \omega dt)^2 - e^{2\mu_2}(dx^2)^2 - e^{2\mu_3}(dx^3)^2$$

Kerr geometry can be derived

$$ds^2 = \rho^2 \frac{\Delta}{\Sigma^2}(dt)^2 - \frac{\Sigma^2}{\rho^2}\left(d\phi - \frac{2aMr}{\Sigma^2}dt\right)^2 \sin^2\theta - \frac{\rho^2}{\Delta^2}(dr)^2 - \rho^2(d\theta)^2$$

- $\rho^2 = r^2 + a^2 \cos^2\theta$
- $\Delta = r^2 - 2Mr + a^2$
- $\Sigma^2 = (r^2 + a^2)^2 - a^2\Delta \sin^2\theta$

The other definition of Kerr metric

$$ds^2 = \left(1 - \frac{2Mr}{\rho}\right) - \frac{4aMr}{\rho^2}\sin^2\theta d\phi dt - \left(\frac{2Ma^2 r\sin^2\theta}{\rho^2} + r^2 + a^2\right)\sin^2\theta d\phi^2 - \rho^2\left(\frac{dr^2}{\Delta} + d\theta^2\right)$$

In asymptotic region (up to $O(\frac{a}{r})$)

- $\rho^2 = r^2 + a^2\cos^2\theta \quad \to \quad r^2$
- $\Delta = r^2 - 2Mr + a^2 \quad \to \quad r^2 - 2Mr$

$$ds^2 \simeq \left(1 - \frac{2M}{r}\right) - \frac{4aM}{r}\sin^2\theta d\phi dt - r^2\sin^2\theta d\phi^2 - r^2\left(\frac{dr^2}{\Delta} + d\theta^2\right)$$

$\to$ **Rotating black hole in the asymptotic region up to $O(\frac{a}{r})$**

In this limit and substitution $2M \to b$ and appropriate g_{tt} make the rotating wormhole model like the static case.

There is mechanism of making wormhole of two black holes by cutting and pasting for the case of Schwarzschild, RN, Kerr(?) black holes and FRW cosmology (Visser)

Otherwise, there are successful models of wormhole with simple substitution.

- static wormhole (Misner and Thorne)

$$ds^2 = -e^{2\Lambda(r)}dt^2 + \frac{dr^2}{1-b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

- inflating wormhole (Roman)

$$ds^2 = -e^{2\Lambda(r)}dt^2 + e^{2\chi t}\left[\frac{dr^2}{1-b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

- wormhole model in FRW cosmology (Kim)

$$ds^2 = -e^{2\Lambda(r)}dt^2 + R(t)\left[\frac{dr^2}{1-b(r)/r+kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

- wormhole with charge (Kim and Lee)

$$ds^2 = -(1+Q^2/r^2)dt^2 + \frac{dr^2}{1-b_0^2/r^2+Q^2/r2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

How about the wormhole with angular momentum ? $\quad\rightarrow\quad$ not certain yet

- **The ring singularity of Kerr metric should not be included in wormhole model which is defined as singularity-free.**

- **The matter part that violates the known energy conditions should be checked in this substitution.**

Scalar Perturbation under Rotating Wormhole

By using $\sqrt{-g} = r^2K^2NL\sin\theta$, where $L^2 = 1/(1-b/r)$, the scalar wave equation

$$\triangle\Phi = \frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu\Phi) = 0$$

In this case we try to variables separately as $\Phi = R(r)\Theta(\theta)e^{im\phi}e^{i\omega t}$.

$$\frac{\omega^2}{N^2} + \frac{2\Omega}{N^2}\omega m - m^2\left(\frac{1}{r^2K^2\sin^2\theta} - \frac{\Omega^2}{N^2}\right) + \frac{1}{r^2K^2NL}\frac{1}{R}\frac{d}{dr}\left(\frac{r^2K^2N}{L}\frac{d}{dr}\right)R +$$
$$\frac{1}{r^2K^2NL\sin\theta}\frac{1}{\Theta}\frac{d}{d\theta}\left(NL\sin\theta\frac{d}{d\theta}\right)\Theta = 0$$

It is very hard to separate variables when N, K, L, and Ω are function of r and θ.

To make the problem simple, we adapt the toy model as $N = K = 1, \quad b(r) = b_0^2/r, \quad \Omega(r)$ such that

$$ds^2 = -dt^2 + \frac{dr^2}{1-b(r)/r} + r^2d\theta^2 + r^2\sin\theta[d\phi^2 - \Omega(r)dt]^2$$

This is the just the stiff rotation of the simplest model. That means that the wormhole does not change its shape under rotation with angular velocity $\Omega(r)$

In this case, the wave equation can be separated as

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)\Theta - \left(\frac{m^2}{\sin^2\theta} - \lambda_{lm}\right)\Theta = 0$$

$$\frac{d^2u}{dr_*^2} - V_l(\omega, m, r, \Omega)u = 0$$

$$R(r) = \frac{u(r)}{r}$$

$$\frac{d}{dr_*} = \sqrt{1 - \frac{b(r)}{r}}\frac{d}{dr}$$

$$V_l = \frac{\lambda_{lm}}{r^2} + \frac{b_0^2}{r^4} - (\omega + \Omega m)^2$$

The separation constant λ_{lm} becomes $l(l+1)$ in the limit of $\omega = 0$. The solution to $\Theta(\theta) = S_{ml}(\theta, 0)$ that is a spheroidal wave function with $\omega\Omega = 0$.

When $r \to \infty$ the solution becomes

$$\frac{d^2u}{dr_*^2} + \omega^2 u \simeq 0$$

which means

$$u \sim e^{\pm i\omega r_*}$$

Summary and Discussion

- We consider the analogy of rotating black hole in the limit of asymptotic region as the rotating wormhole.

- It is the asymptotic model neglecting terms $O((\frac{a}{r})^2)$.

- Simple change of $2M$ into b will not be certain as the Kerr-type wormhole.

- The ring singularity of Kerr metric should not be included in wormhole model which is defined as singularity-free.

- The matter part that violates the known energy conditions should be checked in this substitution.