AdS/dS Correspondence and Twisting S-Branes

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Happy 70th Birthday to Roy!
AdS\(_{d+1}\) space is the surface defined by

\[ x_0^2 - x_1^2 - \cdots - x_d^2 + x_{d+1}^2 = l_{AdS}^2 > 0 \]

After a Wick rotation \(x_{d+1} \rightarrow ix_{d+1}\) and \(l_{AdS} \rightarrow il_{dS}\), the resulting Euclidean space is de Sitter space

\[ -(x_0^2 + x_1^2 + \cdots + x_d^2 + x_{d+1}^2) = l_{dS}^2 > 0 \]

\[ \Rightarrow \quad ds_{dS}^2 = -(dx^0)^2 + \sum_{i=1}^{d+1} (dx^i)^2 \]

In polar coordinates:

\[ r^2 = \sum_{i=1}^{d} (x^i)^2, \quad \frac{x_{d+1} + x_0}{\sqrt{l^2 - r^2}} = e^{\pm \frac{t}{i}} \]
dS space in static coordinates

\[ ds^2_{dS} = - \left( 1 - \frac{r^2}{l^2} \right) dt^2 + \left( 1 - \frac{r^2}{l^2} \right)^{-1} dr^2 + r^2 d\Omega^2_{d-1} \]

One generalizes this to a \((d+1)\)-dim Schwarzschild dS

\[ ds^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega^2_{d-1} \]

where the harmonic function \(f(r)\) is

\[ f(r) = 1 - \frac{r^2}{l^2} - \frac{\mu_{dS}}{r^{d-2}} \]

In the Anti-de Sitter case \(f(r) = k + \frac{r^2}{l^2} - \frac{\mu_{AdS}}{r^{d-2}}\) so that the curvature constant \(k = 0\) or \(\pm 1\).
There are many similarities between AdS and dS spaces, but they still differ in significant ways. Illustration of this fact with some examples is one of the objectives in the first part of this talk.

In Schwarzschild AdS case, i.e., for $k = +1$,

$$\text{Energy} = M_{ADM}, \quad M_{ADM} \propto \mu$$

In de Sitter space, Energy does not mean ADM mass! and so is the case in AdS space when $k \neq 1$

$$E = M_{ADM} + E^{0}_{k}$$
The energy in AdS spacetime

\[ E = \frac{(d - 1)V_{d-1,k}}{16\pi G_{d+1}} \mu + E_k^0 \]

where \( E_k^0 \) depends on regularization and/or asymptotic topology of AdS geometry. From the counter-terms and background subtraction methods, we find, respectively,

\[ E_{k}^0 = \frac{V_{d-1,k}}{16\pi G_{d+1}} \left( -k^2 \delta_{d,2} + \frac{3}{4} k^2 l^2 \delta_{d,4} - \frac{5}{8} k^3 l^4 \delta_{d,6} + \cdots \right) \]

\[ E_{k=-1}^0 = \frac{V_{d-1,k}}{16\pi G_{d+1}} \left( \cdots + \frac{3}{4} l^2 \delta_{d,4} + \frac{20}{27} l^4 \delta_{d,6} + \cdots \right) \]
**Energies in dS spaces**

- Only in AdS$_5$ case, these two results agree, and for $k = +1$, they do agree also in AdS$_3$.

- AdS$_3$/CFT$_2$, AdS$_5$/CFT$_4$ are the two specific examples where gravity-gauge theory duality is much explored.

- The total mass (or quasi-local energy) of an asymptotically de Sitter space is

$$ E = -\frac{(d - 1)V_{d-1,k}}{16\pi G_{d+1}} \mu + E^0_{k=+1} $$

$$ E^0_{k=1} = \frac{V_{d-1}}{16\pi G_{d+1}} \left( \delta_{d,2} + \frac{3}{4} l^2 \delta_{d,4} + \frac{5}{8} l^4 \delta_{d,6} \cdots \right) $$
Energies in AdS and dS spaces

When comparing the quasi-local energies in de Sitter spaces to their AdS counter parts, in none of the dimensions they agree under $l_{AdS}^2 \rightarrow -l_{dS}^2$. In 5D

$$E_{AdS,k=-1} = \frac{3\pi l^2}{64G_5} + \frac{3\pi \mu_{AdS}}{16\pi G}, \quad E_{dS} = \frac{3\pi l^2}{64G_5} - \frac{3\pi \mu_{dS}}{16\pi G}$$

That is, $E_{AdS} = -E_{dS}$ under the rescaling $l_{AdS}^2 \rightarrow -l_{dS}^2$. But it fails in other spacetime dimensions. In fact, in all dimensions, $E_{AdS} = E_{dS}$ under $\mu_{dS} \rightarrow -\mu_{AdS}$; the parameter $\mu$ is actually a measure of the curvature. This matching may not be accidental!

In 5D, the energy $E$ is vanishing in the extremal limit $\mu_{AdS} = -l_{AdS}^2/4$, and in Nariai limit $\mu_{dS} = l_{dS}^2/4$.
AdS $\Rightarrow$ dS: A Heuristic Approach

Consider AdS$_{d+1}$ with the hyperbolic symmetry

$$ds^2 = - (1 + r^2/l^2) dt^2 + \frac{dr^2}{1 + r^2/l^2} + r^2 (d\theta^2 + \sinh^2 \theta d\Omega_{d-1}^2)$$

A complex substitution $r \rightarrow i \rho$, $\theta \rightarrow \tau + i \frac{\pi}{2}$, $t \rightarrow r$
gives a new AdS metric with Lorentzian signature

$$ds^2 = (1 + \rho^2/l^2) dr^2 + \frac{d\rho^2}{1 + \rho^2/l^2} + \rho^2 (-d\tau^2 + \cosh^2 \tau d\Omega_{d-2}^2)$$

$\rho = \text{constant sections have dS}_{d-1}$ topology. Here dS$_{d-1}$ looks like a $(d - 2)$ sphere which starts out infinitely large at $\tau = -\infty$, then shrinks to a minimum finite size at $\tau = 0$ and then grows again to infinite size as $\tau \rightarrow +\infty$
A massive scalar in (A)dS spaces

The KG equation for a massive scalar $\phi$ in AdS

$$(\Box - m_{\text{eff}}^2)\phi = 0, \quad m_{\text{eff}}^2 = m^2 - \frac{(d-1)(d+1)}{4l^2} \gamma$$

$$\phi^\pm(z) = N_m z^{d/2} J_{\mp \nu}(pz), \quad \nu = \sqrt{\left(\frac{d}{2}\right)^2 + m_{\text{eff}}^2 l^2}$$

The solution is normalizable in the regime

$$-\frac{d^2}{4} < m_{\text{eff}}^2 l^2 < 1 - \frac{d^2}{4}$$

No stable scalar with $m_{\text{eff}}^2 < -\frac{d^2}{4}$. While, in de Sitter space, unitarity is preserved only if $\frac{d^2}{4} > m_{\text{eff}}^2 l^2 > -1 + \frac{d^2}{4}$.
S-brane: Why do we buy it?

- A century after the discovery of GR – new solutions continue to be found
  - Static (black holes, p-branes)
  - Time-dependent solutions (S-branes)

- Supergravity S-branes are basically isotropic. Do we know any nice static solutions which are not isotropic?

- Kerr S-brane – obtained by applying a trick of double Wick rotations to static (Kerr) solution – is such example

- I will first review some recently known supergravity S-brane solutions, which apart from string theory interests, e.g., time-dependent tachyon condensation and formation/decay of unstable branes, are of special interest in accelerating cosmologies
D-branes ⇔ S-branes

D-branes are purely static objects, while S-branes are time-dependent configurations.

**R-SYMMETRY**

\[
\begin{align*}
\text{D-brane} & \quad R^{p,1} \times S^{d-p-1} \Rightarrow SO(d-p-1) \\
\text{S-brane} & \quad R^{p+1,1} \times H^{d-p-2} \Rightarrow SO(1, d-p-2)
\end{align*}
\]

For example, when \( D = 4 \)

- **D1-brane:** \( R^{1,1} \times S^2 \) \( SO(2) \)
- **S0-brane:** \( R^{1,1} \times H^2 \) \( SO(1, 2) \)

A solution with the above symmetries generically admits singularities. However, in a twisted space, e.g., \( R^{1,1} \times H^2 \), the solution can be non-singular.
Non-singular S-branes?

S-branes — Apply a trick of double Wick rotations to static solutions. In four dimensions, the S0-brane metric

\[ ds^2_{S0} = \left( 1 - \frac{2m}{\tau} - \frac{e^2}{\tau^2} \right) dz^2 - \frac{d\tau^2}{1 - \frac{2m}{\tau} - \frac{e^2}{\tau^2}} + \tau^2 \left( d\theta^2 + \sinh^2 \theta d\phi^2 \right) \]

is the analytic continuation of Reissner Nordstrom BHs

\[ z \rightarrow i\tau, \quad \tau \rightarrow ir, \quad \theta \rightarrow i\theta, \quad m \rightarrow im \]

where \( z \rightarrow i\tau \) is applied to a Killing direction

Time-symmetric S-branes – no odd powers of \( z \) (\( z \rightarrow -z \)) (ii) suitable condition is imposed on field strength, if present No guarantee that the resulting solutions will be singularity free in general
**Kerr S-branes**

- The 4D Kerr black hole is an axially symmetric solution to the vacuum Einstein equations that describe the spacetime outside of a rotating black hole.

- The Kerr S-brane metric

\[
\begin{align*}
 ds^2_{Kerr-S0} &= -\frac{\Delta}{\Sigma} d\tau^2 + \Delta d\theta^2 + \frac{\Sigma}{\Delta} (dz - a \sinh^2 \theta d\phi)^2 \\
 &\quad + \frac{\sinh^2 \theta}{\Delta} ((\tau^2 + a^2) d\phi + adz)^2, \\
 \Delta &= \tau^2 + a^2 \cosh^2 \theta, \quad \Sigma = \tau^2 + a^2 - 2m\tau
\end{align*}
\]

has no analog of the ring singularity (at \( r = 0, \theta = \pi/2 \)) and no CTCs for \( a > m \), in addition of being geodesically complete. But the stationary region of the geometry may contain CTCs for \( a < m \) as \( g_{\phi\phi} < 0 \) there.
Charged Kerr S-string

A higher dimensional twist gives rise to non-trivial scalar and/or gauge fields in the lower dimensional effective theory. The 5d SKerr solution is

\[
\begin{align*}
 ds_5^2 &= -\frac{\Delta}{\Sigma} d\tau^2 + \frac{\sinh^2 \theta}{\hat{\Sigma}} \left[ \Delta \Sigma + 2m\tau (a^2 + \tau^2) \cos^2 \alpha \right] d\phi^2 \\
 &+ \Delta d\theta^2 + \frac{\tilde{\Sigma}}{\Sigma} dx^2 + \frac{\Delta}{\tilde{\Sigma}} dy^2 - \frac{4ma\tau \cos \alpha \sinh^2 \theta}{\hat{\Sigma}} dx d\phi, \\
 \Sigma &= \tau^2 + a^2 - 2m, \quad \tilde{\Sigma} = \Sigma + a^2 \sinh^2 \theta, \quad \hat{\Sigma} = \tilde{\Sigma} + 2m\tau \cos^2 \alpha, \\
 \Phi &= \ln \frac{\Delta}{\tilde{\Sigma}}, \quad B_{xy} = -\frac{m\tau \sin 2\alpha}{\tilde{\Sigma}}, \quad B_{x\phi} = -\frac{2am\tau \sin \alpha \sinh^2 \theta}{\hat{\Sigma}}.
\end{align*}
\]

In general, a non-singular solution requires \( \sin^2 \alpha < a/m \). In the limit \( \alpha \to 0 \) we find the 4D Kerr solution, and \( \alpha = \pi/2 \) gives a T-dual SKerr solution.
FRW universe and S-branes

Consider intersecting SM2-SM5 branes

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0 \\
\text{SM5} & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\text{SM2} & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\end{array}
\]

The world-volume direction of the S-branes are indicated by a circle, and the remaining directions \((8, 9, 10, 0)\) correspond to our spacetime.

\[
ds^2_{4,E} = g_{\mu\nu,E} \, dx^\mu \, dx^\nu = -a(\tau)^6 \, d\tau^2 + a(\tau)^2 \, d\vec{x}_{3,\epsilon}^2
\]

where \(\epsilon = 0, \pm 1\). The four-dimensional proper time \(t\) is defined by \(dt = a(\tau)^3 \, d\tau\)
Intersecting S-branes solution

The SM2-SM5 intersecting solution is

\[ ds_{11}^2 = K^{1/3} L^{2/3} e^{-2\lambda_1 \tau} \left[ ds_E^2 + \frac{1}{KL} (dy_1^2 + dy_2^2) + \frac{e^{3\lambda_1 \tau}}{L} (dy_3^2 + \cdots + dy_6^2) + \frac{e^{6\lambda_1 \tau}}{K} dy_7^2 \right] \]

\[ K = \cosh(\lambda_2 \tau), \quad L = \cosh(\lambda_3 \tau), \quad a(\tau) = e^{\pm \beta \tau} \]

\[ \beta = \frac{1}{2} \sqrt{6\lambda_1^2 + (\lambda_2^2 + \lambda_3^2)/3} \]

is non-singular, but it does not give an accelerating universe. The situation is same for gravity coupled to a dilaton \( \phi \) and \( m \)-different \( q_A \)-form field strengths.
Interestingly, the supergravity S-brane solutions avoid a ‘no-go’ theorem given for a warped de Sitter type compactification in pure supergravity models, as there exist solutions of 10 or 11d supergravities corresponding to four dimensional inflating universes.

However, unfortunately, S-brane solutions which give rise to four-dimensional inflating universes are often plagued by singularities!

Consider the bosonic sector of 11d supergravity

\[
S = \int d^{11}x \sqrt{-g} \left( \frac{R}{\kappa^2} - \frac{1}{2 \times 4!} F_{[4]}^2 \right) + S_{\text{boundary}}
\]
The metric solution in Einstein conformal frame is

\[ ds_{11}^2 = e^{-\frac{m}{2} \phi(\tau)} g_{\mu\nu} dx^\mu dx^\nu + l_c^2 e^{\phi(\tau)} d\Sigma_{7,\epsilon_1}^2, \]

\[ \phi = \frac{2}{m-1} \ln(KL), \quad a(\tau) = K \frac{m}{2(m-1)} L^{\frac{(m+2)}{6(m-1)}}, \]

\[ K = \begin{cases} 
\frac{1}{m-1} H_0 l_c \ \text{cosech} H_0 |\tau|, & \epsilon_1 = -1, \\
\ e^{\pm H_0 \tau}, & \epsilon_1 = 0, \\
\frac{1}{m-1} H_0 l_c \ \text{sech} H_0 \tau, & \epsilon_1 = +1,
\end{cases} \]

\[ L = \frac{b}{H_0} \sqrt{\frac{m-1}{2m}} \ \cosh \sqrt{\frac{3m}{m+2}} H_0 \tau, \quad *F_4 = b \ \text{vol} (\Sigma_{m,\epsilon_1}) \]

\[ \epsilon_1 = -1 \] solution gives a transient acceleration, even if the flux parameter \( b \) is zero. Unlike with a twist, \( b > 0 \) solution does not reduce to the one with \( b = 0 \).
Summary

- dS/CFT may be mapped to a somewhat unconventional AdS/CFT duality when the latter is defined on an \((d - 1)\)
  dimensional Euclidean AdS (hyperbolic) space

- SKerr solutions are free of curvature singularities when suitable condition is imposed on the twist parameter, and field strengths, if present.

The analogue time-dependent supergravity S-Kerr solutions may have important implications for cosmology

- better than what one expects at the outset – notably, a non-trivial twist can generate a less steep scalar potential, allowing eternally inflating flat or open universes; an interesting example is \(R^{3,1} \times (H^2 \times S^1) \times M^2 \times M^2\)