Four decades of black hole uniqueness theorems
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Outline

- Israel's groundbreaking 1967 theorem - the static vacuum case.
- Issues raised by Israel's theorem.
- How these issues have been dealt with:

  (a) sketch of developments by decades, 1970's, 1980's, 1990's- 2000's,

  (b) selected topics explored in more detail. Fermionic fields are not discussed.
- The current status of the static/stationary a.f. vacuum and electrovac results in four dimensions.
- Caveat: "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality" - Albert Einstein.

Israel's first 1967 theorem - the static vacuum problem

Event Horizons in Static Vacuum Space-Times (W. Israel Phys. Rev. 164, 1776, 1967)

This dealt with a four dimensional static space-time satisfying Einstein's vacuum field equations,

$$R_{\alpha\beta} = 0.$$

The space-time is static, i.e. it admits a time-like hypersurface orthogonal Killing vector field, K^α,

$$K^\alpha K_\alpha < 0, K_{[\alpha}\nabla_\beta K_{\gamma]} = 0,$$

and an adapted coordinate system (t, x^a), such that

$$\begin{aligned} ds^2 &= -V^2 dt^2 + g_{ab} dx^a dx^b, \\ K^\alpha &= (1, 0, 0, 0); V_{,t} = g_{ab,t} = 0. \end{aligned}$$

Israel's assumptions in his 67 paper:

In a static space-time let Σ be any spatial hypersurface t=const., maximally consistent with $K^\alpha K_\alpha < 0$. We consider the class of static fields such that the following conditions are satisfied on Σ.

(a) Σ is regular, empty, non-compact and "asymptotically Euclidean", i.e. the metric (in suitable coordinates) has the asymptotic form

$$\begin{aligned} g_{ab} &= \delta_{ab} + O(r^{-1}), \quad g_{ab,c} = O(r^{-2}), \\ V &= 1 - mr^{-1} + \eta; \; m \text{ const.}; \; \eta = O(r^{-2}), \\ \eta_{,a} &= O(r^{-3}), \eta_{,ab} = O(r^{-4}); \; \text{as } r = (\delta_{ab}x^a x^b)^{1/2} \to \infty. \end{aligned}$$

(b) The equipotential surfaces in Σ, V =const.> 0, are regular, simply connected closed 2-spaces.

(c) The invariant ${}^{(4)}R_{\alpha\beta\gamma\delta}.{}^{(4)}R^{\alpha\beta\gamma\delta}$ formed from the four dimensional Riemann tensor is bounded on Σ.

(d) If V has vanishing lower bound on Σ, the intrinsic geometry (characterized by ${}^{(2)}R$) of the 2-spaces $V = c$ approaches a limit as $c \to 0^+$ corresponding to a 2-space of finite area.

Israel's 67 Theorem:
The only static space-time satisfying (a), (b), (c) and (d) is Schwarzschild's spherically symmetric vacuum solution.

$$ds^2 = -(1 - 2mr^{-1})dt^2 + (1 - 2mr^{-1})^{-1}dr^2 + r^2 d\Omega^2,$$
$$2m < r < \infty.$$

(electrovac extension by Israel, 68)

Issues raised by Israel's theorem

i. The equipotential condition (b).

ii. The assumption of spherical topology of the two dimensional surface of the black hole.

iii. The assumption that the black hole horizon had only one connected component.

iv. The effect of the inclusion of matter fields and/or a cosmological constant.

v. The corresponding situation for stationary and rotating black holes.

vi. The relationship between the "reduced Riemannian" problem, the Lorentzian 4- geometry & the horizon.

vii. The role of the dimension of space-time.

viii. The limits of uniqueness (and stability).

ix. The effect of changing the field equations.

x. Sharpness, rigour, differentiability, analyticity.

- 1970's : Laying the foundations

- Assumption (b) on the equipotential surfaces in Israel's theorem was removed & the geometrical structure of the proof, in particular the significance of the conformal 3-geometry, was elucidated.

  (Muller Zum Hagen, Robinson & Seifert, 73, Israel's static electrovac theorem also extended, 74.)

- The non-existence, in the axially symmetric, static vacuum case, of equilibrium configurations of more than one black hole (Muller Zum Hagen & Seifert, 73) and of black holes and massive bodies which do not surround or partially surround a black hole were demonstrated. (Gibbons, 74)

  [Investigation of Weyl metrics corresponding to axially symmetric static black holes with external bodies were continued later. (e.g. Geroch & Hartle, 82).]

- The existence of static, charged multi-black hole systems was noted.

  (Hartle &Hawking: the Majumdar - Papapetrou solutions, 72).

  Their uniqueness in the axially symmetric case was argued. (Gibbons, 80).

- The detailed investigation of the existence and uniqueness of black holes with scalar fields was initiated.

  (static - Chase, 70; Beckenstein, 72, 74, 75; Hawking 72).

1970's: laying the foundations continued

- Foundations for future investigations were established. In a seminal paper published in 1972 Hawking included basic global results for four dimensional, asymptotically flat, stationary black hole systems. In this paper Hawking showed, that under certain conditions the topology of two dimensional cross sections of the horizon was spherical (possibly toroidal?). *A Rigidity Theorem*, which relates the logically distinct notions of Killing and event horizons, was also introduced in this paper. This crucial theorem underlies the reduction of the uniqueness problems to two distinct types of Riemannian boundary value problems: in the static case - to a three dimensional Riemannian problem; in the stationary case - to axial symmetry and a two dimensional Riemannian problem. These results have been the subject of much subsequent work and have provided the basis for many future developments. For example the *Staticity Theorem* has been firmly established in the vacuum and electrovac case. That is, it has been proven that non-degenerate stationary black holes with event horizons which are the union of Killing horizons of the asymptotically time like Killing vector field are static. (vacuum, electrovac... extension completed by Sudarsky & Wald 93). In addition, around the same time, the basic and extensive machinery for investigating stationary axi-symmetric black holes was established by Carter. This included the reduction of the axially symmetric, stationary, vacuum & electrovac, uniqueness problem to a two dimensional boundary value problem for systems of elliptic p.d.e.'s. (Hawking & Carter: summarized in the Les Houches lectures, 73 and in the 73 monograph by Hawking & Ellis).

  While a number of problems associated with this agenda-setting work have now been satisfactorily resolved the analyticity assumptions in the Rigidity Theorem have yet to be satisfactorily dealt with. These matters have been extensively discussed in, for example, the reviews listed below, so this talk will focus on the reduced Riemannian problems.

- No-hair theorems for asymptotically flat, axisymmetric, stationary, vacuum and electrovac black holes with non-degenerate horizons, leading to the proof of the uniqueness of the Kerr solution, were exhibited. (Carter, 71; Robinson 74, 75, 75) .

A 1970's proof of Israel's theorem

(4d, vacuum, connected, without the equipotential or spherical topology assumptions, Robinson 77)

Einstein's field equations (in d>2 dimensions) are

$$^{(d)}R_{\alpha\beta} = 8\pi(T_{\alpha\beta} - \frac{1}{d-2}g_{\alpha\beta}T) + \frac{2}{d-2}\Lambda g_{\alpha\beta}$$

For static metrics in static coordinates, the vacuum, $\Lambda = 0$, field equations are

$$ds^2 = -V^2dt^2 + g_{ab}dx^a dx^b,$$

$$^{(d)}R_{tt} \equiv V D^a D_a V = 0$$
$$^{(d)}R_{ta} \equiv 0$$
$$^{(d)}R_{ab} \equiv R_{ab} - V^{-1}D_a D_b V = 0.$$

The black hole boundary conditions as formulated on Σ, $t = const.$ which is regular, and where $0 < V < 1$, are:
(a) asymptotic flatness (as $V \to 1$)
(b) regularity of the horizon H (as $V \to 0$)
$\partial\Sigma$ regular, connected, compact geodesic 2-surface in $\overline{\Sigma}$ and

$$W \equiv W = -\frac{1}{2}\nabla_{[\alpha}k_{\beta]}\nabla^{\alpha}k^{\beta} = g^{ab}V_{,a}V_{,b} = W_0 = const.on\ \partial\Sigma.$$

The latter constant is the square of the surface gravity, and is necessarily non-zero (i.e. the horizon is non-degenerate) in this connected horizon case.

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Let R_{abc} be the Cotton tensor for g_{ab},

$$R_{abc} = D_b(R_{ac} - \frac{1}{4}Rg_{ac}) - D_c(R_{ab} - \frac{1}{4}Rg_{ab}).$$

From the vacuum field equations:

$$\begin{aligned} D^a D_a V &= 0, && \text{(i)} \\ D_a(V^{-1}D^a W) &= 2V R_{ab}R^{ab}, && \text{(ii)} \\ R_{abc}R^{abc} &= 4V^{-4}W D_a D^a W - 4V^{-5}W D^a W D_a V - \\ &\quad - 3V^{-4}D_a W D^a W, \end{aligned}$$

and (iii)

$$\begin{aligned} & D_a(pV^{-1}D^a W + qW D^a V) \\ & \qquad = \frac{p}{4}V^3 W^{-1} R_{abc}R^{abc} + \\ & \qquad + \frac{3}{4}V^{-1}W^{-1}p[D_a W + 8WV(D_a V)(1-V^2)^{-1}]^2, \end{aligned}$$

where

$$\begin{aligned} p(V) &= (cV^2 + d)(1-V^2)^{-3} > 0; \\ q(V) &= -2c(1-V^2)^{-3} + 6(cV^2+d)(1-V^2)^{-4}. \end{aligned}$$

The divergence in (iii) is everywhere regular and non-negative on Σ. When $W = 0$, $R_{abc} = D_a W = 0$.

Integrating (i)-(iii) over Σ gives

i) $4\pi m = W_0^{1/2} S_0$. (this is the Smarr formula from which non-degeneracy necessarily follows).

ii) $W_0^{1/2} \int_H ({}^{(2)}R) dS \geq 0$;

since by the Gauss-Bonnet theorem, $\int_H ({}^{(2)}R) dS = 8\pi(1-p)$, $p = 0, 1, 2...$, it follows that $p = 0$, and the topology of H must be spherical.

iii)

$$\text{with } c = -1, d = +1; \qquad W_0 S_0 \leq \pi,$$

$$\text{with } c = 1, d = 0; \qquad W_0^{3/2} S_0 \geq \frac{\pi}{4m}.$$

It follows from these inequalities and the equality i) that R_{abc} must be zero and hence the 3-geometry must be conformally flat, and also that $W = \frac{(1-V^2)^4}{16m^2}$. It is then straightforward to show that the space-time must be Schwarzschild.

1980's: Systematization and new beginnings

- The introduction of new techniques.

  (a) In the static case, the positive energy theorem, proven in 1979, was used to deal with non-connected, non-degenerate horizons. (Bunting & Masood-ul-Alam, 87)

  (b) In the stationary, axially symmetric case, a sigma model/ harmonic maps approach was introduced to deal with the reduced Riemannian problem. The Kerr uniqueness proof was extended to the electrovac case and the uniqueness of the Kerr-Newman solution was demonstrated.(Bunting 83, Mazur, 82).

- Black holes with hair started to be found in some Einstein-Yang-Mills systems.

- Kaluza-Klein & higher dimensional black hole solutions were found and studied e.g.

  Generalization of Kerr (Myers & Perry, 86), Einstein-Maxwell-dilaton black hole (Gibbons, 82; Gibbons & Maeda, 88), generalization of Majumdar-Papapetrou (Myers, 86).

Non-existence of static multiple black holes - horizon not assumed connected

The exterior Schwarzschild solution is the maximally extended static, vacuum, asymptotically flat space-time with *non-degenerate*, regular black hole boundary. (Bunting & Masood-ul-Alam, 87)

They proved $\Sigma, (t = const)$, must be conformally flat by a new technique, as follows.

(i) They used a corollary to the positive mass theorem (Schoen & Yau 79, Witten 81)

Let (N, γ) *be a complete Riemannian 3-manifold which is asymptotically Euclidean (topology plus fall off,* $\gamma = (1 + \frac{2m}{r})\delta + ..)$. *If the scalar curvature of* γ *is non-negative and* $m = 0$, *then* (N, γ) *is isometric to* (R^3, δ).

(ii) They turned (Σ, g) into an asymptotically Euclidean, complete, Riemannian manifold with zero scalar curvature by

(a) making a conformal transformation $g \rightarrow \mp\gamma = \frac{1}{16}(1 \mp V)^4 g$ on 2 copies of (Σ, g) so that $(\Sigma, +\gamma)$ is asymptotically Euclidean with $m = 0$ and $(\Sigma, -\gamma)$ "compactifies the infinity",

and by

(b) gluing the 2 copies of Σ along their boundaries to form (N, γ).

Then, (N, γ) must be flat by the corollary above and therefore (Σ, g) must be conformally flat.

Virtues included: didn't assume horizon connected, more easily generalizable to complicated matter systems than older proofs, also generalizable to higher dimensions since it didn't use the 3 dimensional conformal tensor - the Cotton tensor.

Stationary, axially symmetric, asymptotically flat black holes, sigma models & Kerr-Newman

Carter showed that coordinates exist on the (regular) domain of outer communication of a stationary axi-symmetric, asymptotically flat, electrovac black hole system with a connected, non-degenerate (regular) event horizon such that

$$ds^2 = -\frac{\rho^2}{X}dt^2 + X(d\phi + Adt)^2 + \frac{e^{2h}\mu^2(x^2 - y^2)}{X}d\sigma^2,$$
$$\rho^2 = \mu^2(x^2 - 1)(1 - y^2),\ \mu^2 = M^2 - (J/M)^2 - Q^2.$$
$$d\sigma^2 = \frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2}.$$

Carter also showed that the black hole uniqueness problem could be reduced to a boundary value problem, on a 2 dimensional manifold with coordinates (x, y) and metric $d\sigma^2$. In the *vacuum* case the relevant field equations constitute an elliptic system with (complex) independent variable, $E = -X + iY$ which determined A and X and hence the whole metric. The Lagrangian for the vacuum differential equations is

$$L = \frac{\nabla E.\nabla \overline{E}}{(E + \overline{E})^2};$$

with field equations (after Ernst):

$$\nabla(\frac{\nabla E}{(E + \overline{E})^2}) + \frac{2\nabla E.\nabla \overline{E}}{(E + \overline{E})^3} = 0.$$

The vacuum black hole boundary conditions computed by Carter are:

asymptotic flatness conditions as $x \to \infty : Y = 2Jy(3 - y^2) + O(x^{-1})$, $X = c^2x^2(1 - y^2) + O(x^{-1})$;

regular symmetry axis conditions as $y \to \pm 1$:

$X \& Y$ well behaved and $X, \partial_x Y, \partial_y Y = O(1 - y^2)$,

$X^{-1}\partial_y X = -2y(1 - y^2)^2 + O(1 - y^2)$.

regular horizon as $x \to 1 : X \ \& \ Y$ well behaved, $X, X^{-1}, \partial_x Y, \partial_y Y = O(1)$.

The 1975 proof of the uniqueness of the Kerr black hole was obtained by using two solutions E_1 and E_2 to construct a generalized Green's identity of the form *divergence = positive terms mod field equations.* This was integrated over the 2-dimensional manifold. Stokes' theorem and the boundary conditions were then used to show that the integral of the left hand side was zero. Consequently all the postive terms on the right hand side had to be zero. This implied that $E_1 = E_2$, and hence uniqueness followed. The electrovac uniqueness problem was formally similar, but technically more complicated, there were more equations and more independent variables. It was expected that the vacuum proof would be extendable to the electrovac case, but that it would be a lengthy, and somewhat unsatisfying matter, just to construct the appropriate identity. It had long been realized that the structure of the Lagrangians for these systems might well play an important role in the uniqueness theorems. The latter had in fact been explicitly indicated in a re-formulation of Carter's vacuum no-hair theorem (Robinson, 75). However the role of the Lagrangian was not properly understood and exploited until the (independent) work of Bunting & Mazur. They observed that the theory of harmonic maps, or sigma models, over a two dimensional manifold could be used to reformulate the Kerr uniqueness proof in a systematic way. This made the extension to a proof of the uniqueness of the Kerr-Newman solution relatively straightforward. Their approaches provided a rationale lacking in the previous somewhat ad hoc constructions, and enabled the possibilities for generalizations - or not, to be considered within a well-understood framework.

A brief outline of Mazur's approach (for a detailed review see Mazur hep-th/0101012).

Recall that the theory of non-linear sigma models includes the study of harmonic maps from a Riemannian space M (x^A) to a Riemannian coset space N=G/H, where G is a non-compact group and H is a maximally compact sub-group. The harmonic maps correspond to solutions of Euler-Lagrange equations. Mazur related the sigma model formalism to the Lagrangian formalism for the vacuum and electrovac field equations making use of earlier work on the stationary field equations by Kinnersley and others. Working within the sigma model context he was able to construct a Green's identity, of the type mentioned above, when the Riemannian symmetric space N, with non-compact isometry group G, had a non-positive sectional curvature.

Appropriate application of this result gave the generalized Green's identities needed to prove the black hole uniqueness results. For vacuum & Kerr uniqueness: N=SU(1,1)/U(1): For electrovac and Kerr-Newman uniqueness: N=SU(2,1)/S(U(2)×U(1)).

- The new approach to static black holes was applied to prove uniqueness in the non-degenerate electrovac and Einstein-Maxwell-dilaton systems without assuming connectedness of the event horizon. (e.g. Masood-ul-Alam (92, 93), Mars & Simon gr-qc/0105023)

  The existence of non-connected regular *stationary* axi-symmetric vacuum black holes is an open question.

  There are some related results: non-existence of 2 identical black holes [Kreuzer 1999, Neugebauer 2000], general results, [Weinstein. gr-qc/9412036, dg-ga/9509003].

- "Mathematical relativity" & rigorous approach to "theorems", filling in holes in old proofs, sharpening and extending results. The development of a programme of classification of static & stationary solutions. (e.g the continuing work of Wald et al & Chrusciel,"No Hair" Theorems - Folklore, Conjectures, Results. gr-qc/9402032).

  In particular

  (a) Staticity theorem completed at least up to electrovac, progress on analyticity/ horizon question made. (Sudarsky & Wald; Chrusciel gr-qc/0402087)

  (b) Proof of the static vacuum uniqueness theorem extended to allow the possibility that the horizon had $N > 1$ components, including *degenerate* ones. Similar, although not quite so complete results have been obtained in the electrovac case. (Chrusciel, see below)

- Many different matter sources, black holes with hair and $\Lambda \neq 0$,considered. String/brane motivated new field equations introduced.

- Higher dimensional uniqueness theorems proven.

Higher dimensional asymptotically flat black holes (d>4)

Motivation for consideration from the conjectures that black hole production may occur in high energy experiments (TeV gravity).

A. Higher dimensional static cases:

Example of higher dimensional static black hole family:

Static electrovac black hole metric, $Reissner-Nordstrom, d > 3$ (Tangherlini, Nuovo Cimento,27, 636, 1963.):

$$ds^2 = -V^2dt^2 + V^{-2}dr^2 + r^2d\Omega^2_{d-2}$$

$$V = (1 - \frac{C}{r^{d-3}} + \frac{D^2}{r^{2(d-3)}})^{1/2},$$
$$F = -\frac{\partial h}{\partial r}dt \wedge dr, \quad h = \pm[\frac{d-2}{8\pi(d-3)}]^{1/2}\frac{D}{r^{d-3}}$$
$$Mass = \frac{C(d-2)A_{d-2}}{16\pi}, \quad Q = \pm D[\frac{(d-3)(d-2)}{8\pi}]^{1/2}.$$

The global structure is similar to the four dimensional case. Stability analysis, similar to four dimensional analyses of Regge - Wheeler-Vishveshwara, has been carried out. There are no regular static pertubations. (Ishibashi & Kodama, hep-th/0305185).

Four dimensional uniqueness results for static black holes have been extended to uniqueness of d>4 Schwarzschild, Reissner-Nordstrom, e.m./dilaton, sigma model.... The method of Bunting & Massod ul Alam is used to prove conformal flatness of the spatial part of the metric. This, plus knowledge of the conformal factor, allows uniqueness of (the known) spherically solutions to be derived fairly directly. (Hwang, Geometriae Dedicata, 71, 5, 98; Gibbons, Ida & Shiromizu, gr-qc/0203004, hep-th/ 0206136; Rogatko hep-th/0207187, 0302091, 0406041)

The theorems depend on higher dimensional positive energy theorems. (e.g. Dai, math-ph/0406006)

B. Higher dimensional stationary case:

Example of higher dimensional stationary, asymptotically flat, vacuum black hole family:

Myers-Perry 1986 generalization of the Kerr solution admits, in general, [(d-1)/2] non-zero spins. Here the single spin case is exhibited:

$$ds^2 = -dt^2 + \frac{\mu}{r^{d-5}\rho^2}(dt + a\sin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 +$$
$$+ \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\Omega^2_{(d-4)}.$$
$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 + a^2 - \frac{\mu}{r^{(d-5)}},$$
$$M = \frac{(d-2)\Omega_{(d-2)}}{16\pi}\mu, \quad J = \frac{2Ma}{(d-2)}.$$

1. For d=4 reduces to Kerr where a horizon exists if $| J | \leq M^2$.

2. d>4, 3 Killing vector fields.

3. For d=5 there is a horizon if $\mu > a^2$ and no horizon if $\mu \leq a^2$.

4. For d>5 there is always a horizon, independent of the magnitude of a i.e. for arbitrarily large spin.

(Suggested instability - Emparan & Myers, hep-th/0308056)

5. For d odd, black holes with negative mass -causality violations.

Non-Uniqueness of the rotating black hole

However, in five dimensions there is also an asymptotically flat, stationary, vacuum solution with a horizon of topology $S^1 \times S^2$: a black ring, characterized by mass M and spin J. It has 3 Killing vector fields. The five dimensional Myers-Perry solution with rotation in a single plane (and horizon of topology S^3) can be obtained as a branch of the same family of solutions. There exist Myers-Perry & Emparan-Reall black holes and black rings with the same values of M and J. They can be distinguished by their topology and by their mass dipole measured at infinity. Emparan & Reall (hep-th/0110260)

In this case of the Emparan-Reall family however, there is no static and spherically symmetric limit black hole. Pertubations analysis off spherically symmetric vacuum solution suggests that the Myers-Perry solutions are the only regular black holes near the static limit (Kodama hep-th/0403239).

Question: Is there Uniqueness/ no hair subject to extra conditions, e.g. fixed horizon topology, stability (Kol, hep-th/0208056), or fixed higher multipole moments(Tomizawa et al, gr-qc/0405134)?

Recent extension of Kerr uniqueness to d=5:

In five dimensions, assuming *two* commuting *rotational* Killing vectors in addition to the stationary Killing vector field, and the sphericity of the horizon topology, the vacuum black hole is uniquely characterized by the mass and a pair of angular momenta, i.e. by the Myers Perry class of solutions.

Method of proof: Carter/Mazur type formulation leading to a sigma model, SL(3,R)/SO(3), boundary value problem.

Problems with extension to higher dimensions: "An n-dimensional space-time admitting (n-3) commuting Killing vector fields is always describable by a non-linear sigma model (Maison). However the n-dimensional Myers-Perry black hole space-time has only [(n-1)/2] commuting space-like Killing vector fields."

(Morisawa & Ida, gr-qc/0401100)

Current state of uniqueness results for vacuum and electrovac in 4 dimensions

1. Most complete formulation of Israel's theorem for static, vacuum black holes in four dimensions

(Chrusciel, gr-qc/9809088)

Let (M,g) be a static solution of the vacuum Einstein equations with defining Killing vector K. Suppose that M contains a connected space-like hypersurface Σ the closure $\overline{\Sigma}$ of which is the union of asymptotically flat ends and of a compact interior, such that:

1. We have $g_{\mu\nu}K^{\mu}K^{v} < 0$ on Σ.

2. The topological boundary $\partial\Sigma = \overline{\Sigma}\backslash\Sigma$ of Σ is a non-empty topological manifold with $g_{\mu\nu}K^{\mu}K^{v} = 0$ on $\partial\Sigma$.

Then Σ is diffeomorphic to R^3 minus a ball, so that it is simply connected, it has only one asymptotically flat end, and its boundary $\partial\Sigma$ is connected. Further there exists a neighbourhood of Σ in M which is isometrically diffeomorphic to an open subset of the Schwarzschild space-time.

2. A recent formulation of Israel's theorem for static, electrovac space-times in four dimensions

(Chrusciel gr-qc/9810022; see also Rubak, 88, Simon 92 , Heusler 94, 97)

Let (M, g, F) be a static solution of the Einstein-Maxwell equations with defining Killing vector K. Suppose that M contains a connected and simply connected space-like hypersurface Σ, the closure $\overline{\Sigma}$ of which is the union of an asymptotically flat end and of a compact interior such that:

i. We have $g_{\mu\nu}K^\mu K^v < 0$ on Σ.

ii. The topological boundary $\partial\Sigma = \overline{\Sigma}\backslash\Sigma$ of Σ is a non-empty topological manifold with $g_{\mu\nu}K^\mu K^v = 0$ on $\partial\Sigma$.

Then:

1. If $\partial\Sigma$ is connected then Σ is diffeomorphic to R^3 minus a ball. Moreover there exists a neighbourhood of Σ in M which is isometrically diffeomorphic to an open subset of the (extreme or non-extreme) Reissner-Nordstrom space-time.

2. If $\partial\Sigma$ is not connected, and if for all $i, j : Q_i Q_j \geq 0$, where Q_i is the charge of the i-th connected component of $\partial\Sigma$ that intersects the degenerate horizons, then Σ is diffeomorphic to R^3 minus a finite union of disjoint balls. Moreover the space-time contains only degenerate horizons, and there exists a neighbourhood of Σ in M which is isometrically diffeomorphic to an open subset of the standard Majumdar-Papapetrou space-time.

$$ds^2 = -V^2 dt^2 + V^{-2}\delta_{ab}dx^a dx^b;\ V^{-1} = 1 + \Sigma_i \frac{m_i}{r_i},$$

$$r_i = [(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2]^{1/2}.$$

$r_i = 0$ event horizon cpt. $A_i = 4\pi m_i^2, \mid Q_i \mid = m_i$.

3. The generally agreed stationary uniqueness results (as before)

The non-degenerate Kerr black holes satisfying $m^2 > a^2$

$$
\begin{aligned}
ds^2 = \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 - (\frac{\Delta - a^2 \sin^2 \theta}{\Sigma}) dt^2 - \\
- (\frac{2a \sin^2 \theta (r^2 + a^2 - \Delta}{\Sigma}) dt d\varphi + \\
+ \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \sin^2 \theta d\varphi^2; \\
where\ \Sigma = r^2 + a^2, \Delta = r^2 + a^2 - 2mr. \\
0 \leq a < m; \\
r_+ = m + (m^2 - a^2)^{1/2} < r < \infty.
\end{aligned}
$$

exhaust the family of non-degenerate stationary axisymmetric vacuum connected black holes.

Similarly for Kerr-Newman black holes satisfying $m^2 > a^2 + P^2 + Q^2$.

(The condition of axisymmetry can be removed if Hawking's Rigidity theorem can be completed e.g. by removing the analyticity requirements.)

Some surveys & reference sources:

- W Israel, *Dark stars: the evolution of an idea,* pp 199-277 in *300 years of gravitation*, eds. S Hawking & W Israel (Cambridge University Press, 1987)
- M Heusler, *Black Hole Uniqueness Theorems,* (Cambridge University Press, 1996)
- M Heusler*, Stationary Black Holes: Uniqueness and beyond,* Living Rev. Relativity 1, (1998), http://www.livingreviews.org/Irr-1998-6
- B Carter*, Has the black hole equilibrium problem been solved?,* gr-qc/9712038
- P Chrusciel, *Black holes,* gr-qc/0201053